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AN INTRODUCTION TO THE CALCULUS

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BY

CLEMENT V. DURELL, M.A.

SENIOR MATHEMATICAL MASTER, WINCHESTER COLLEGE

AND

R. M. WRIGHT, M.A.

SECOND MASTER, WINCHESTER COLLEGE

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PREFACE

The substance of this book has already appeared as a section of Part II. of the *Elementary Algebra* by the same authors. It is here reprinted with minor modifications for the convenience of those teachers who require a concise first course in the Calculus in a separate form. The scope of the present volume meets the requirements of the Oxford and Cambridge Joint Board School Certificate Examination.

R. M. W.
C. V. D.

April, 1926.

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CHAPTER I.

FUNCTIONS OF ONE VARIABLE.

A. GRAPHICAL METHODS.

Example I. $ABCD$ is a rectangle such that $AB=4''$, $BC=8''$; P, Q are points on BC, CD, such that

$$CQ=\tfrac{1}{2}BP=x \text{ inches.}$$

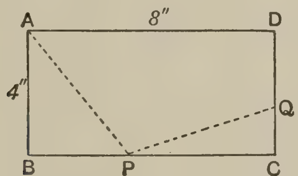

FIG. 1.

Express the area of $APQD$ in terms of x. Represent this function graphically: and find from the graph the length of CQ when the area of $APQD$ is 24 sq. inches.

Check the result by algebra.

(i) The area of $ABCD=4\times 8=32$ sq. inches.

$$CQ=x, \quad BP=2x; \quad \therefore\ PC=8-2x:$$

$\therefore$ area of triangle $ABP=\frac{1}{2}\times 4\times 2x=4x$ sq. inches,

area of triangle $PCQ=\frac{1}{2}(8-2x)\times x=x(4-x)$ sq. inches;

$$\begin{aligned}\therefore \text{ area of } APQD &= 32-4x-x(4-x)\\ &= 32-4x-4x+x^2\\ &= 32-8x+x^2 \text{ sq. inches.}\quad \textit{Answer.}\end{aligned}$$

(ii) To represent $x^2-8x+32$ by a graph.

Since CQ does not exceed CD, x is not greater than 4.

Construct a table of values for x from 0 to 4.

x	0	1	2	3	4
x^2 $-8x$ 32	0 0 32	1 -8 32	4 -16 32	9 -24 32	16 -32 32
$x^2-8x+32$	32	25	20	17	16

Plotting these, we obtain the required graph.

(iii) From the graph we see that the function equals 24 if $x=1{\cdot}17$;

$$\therefore\ CQ=1{\cdot}17'' \text{ when the area of } APQD \text{ is 24 sq. in.}$$

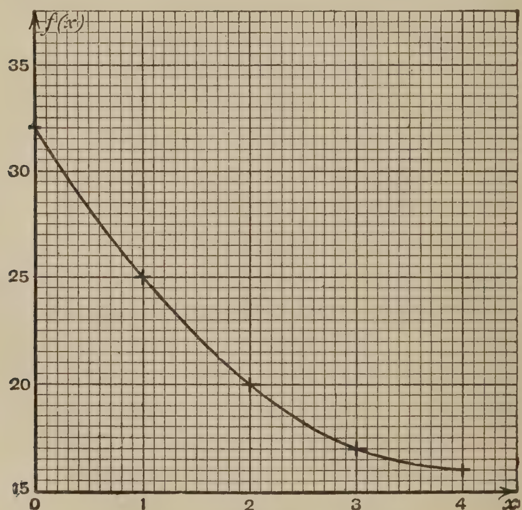

FIG. 2.

(iv) To solve by algebra, we take the equation

$$x^2-8x+32=24;$$

$$\therefore\ x^2-8x+8=0;$$

$$\therefore\ x=\frac{8\pm\sqrt{64-32}}{2}=\frac{8\pm\sqrt{32}}{2}$$

$$=\frac{8\pm5\cdot657}{2}=\frac{13\cdot657}{2}\quad\text{or}\quad\frac{2\cdot343}{2}$$

$$=6\cdot828\quad\text{or}\quad1\cdot171.$$

The value 6·828 is excluded by geometrical considerations;

$$\therefore\ x=1\cdot171.$$

EXERCISE I. a.

1. Fill in the gaps in the following table, and so construct a table of values of the function $4+2x-x^2$ for values of x from -2 to $+4$:

x	-2	-1	0	1	2	3	4
4 $+2x$ $-x^2$	4 -4 -4					4 6 -9	
$4+2x-x^2$	-4					1	

Draw accurately the graph of this function, and from your graph answer the following questions:

(i) For what values of x does $4+2x-x^2=0$?

(ii) Between what values of x is $4+2x-x^2$ positive?

(iii) What is the maximum value attained by the function $4+2x-x^2$, and for what value of x does the function have its maximum value?

(iv) For what values of x does $4+2x-x^2=1\frac{1}{2}$? Solve the equation $2(4+2x-x^2)=3$.

(v) If $2x-x^2=-2$, what is the value of $4+2x-x^2$? For what values of x does $4+2x-x^2$ have this value? Solve the equation $2x-x^2=-2$.

(vi) Solve the equation $4+2x-x^2=-1$ from the graph, and solve the equation $x^2-2x-5=0$ by formula. Compare the results.

2. Construct a table of values of the function $\frac{1}{5}(2x^2+2x-3)$ for values of x from -4 to $+4$ as follows:

x	-4	-3	-2	-1	0	1	2	3	4
$2x^2$ $2x$ -3	32 -8 -3						8 4 -3		
$2x^2+2x-3$	21						9		
$\frac{1}{5}(2x^2+2x-3)$	$4{\cdot}2$						$1{\cdot}8$		

Draw accurately the graph of this function, and from your graph answer the following questions:

(i) For what values of x is $2x^2+2x-3=0$?

(ii) Between what values of x is $2x^2+2x-3$ negative?

(iii) What is the minimum value attained by the function $\frac{1}{5}(2x^2+2x-3)$, and for what value of x does the function have its minimum value?

(iv) For what values of x does $\frac{1}{5}(2x^2+2x-3)=1$? Solve the equation $2x^2+2x-3=5$.

(v) If $2x^2+2x=7$, what is the value of $2x^2+2x-3$ and of $\frac{1}{5}(2x^2+2x-3)$? For what values of x does $\frac{1}{5}(2x^2+2x-3)$ have this value? Solve the equation $2x^2+2x=7$.

(vi) Solve the equation $x^2+x=1$ from the graph. Check your result by solving the equation by the formula.

3. Draw the graph of the function $\frac{1}{10}(x^3-4x)$ for values of x from -4 to $+4$, and use it to answer the following questions:

(i) For what positive value of x is the function a minimum?

(ii) What is the maximum value of the function for negative values of x?

(iii) For what values of x is the function negative?

(iv) For what values of x does the function equal $0{\cdot}2$?

(v) Solve the equations:

(*a*) $\frac{1}{10}(x^3-4x)=0{\cdot}1$; (*b*) $x^3-4x=1$; (*c*) $x^3-4x+1=0$.

4. Construct a table of values of the function

$$\frac{1}{5}(x^3-3x^2-4x-6)$$

for values of x from -3 to $+5$, and draw the graph of this function.

From your graph answer the following questions :

(i) For what values of x does $x^3 - 3x^2 - 4x - 6 = 0$?

(ii) For what range of values of x is $x^3 - 3x^2 - 4x - 6$ positive ?

(iii) What is the smallest value of the function

$$\tfrac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for positive values of x ? For what positive value of x does it have the smallest value ?

(iv) What is the largest value of the function for negative values of x, and for what negative value of x does it have its largest value ?

(v) Solve the equation $x^3 - 3x^2 - 4x - 6 = 7$ from the graph.

(vi) Solve the equations : (a) $x^3 - 3x^2 - 4x + 4 = 0$;
(b) $x^3 - 3x^2 - 4x + 19 = 0$;
(c) $x^3 - 3x^2 - 4x - 16 = 0$.

(vii) Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ from the graph, and also by factorising $x^3 - 3x^2 - 4x + 12$.

5. Draw a graph of the function $\dfrac{1}{x^2}$ for values of x from -2 to $-0{\cdot}5$ and from $0{\cdot}5$ to 3.

(i) With the same axes, and to the same scale, draw the graph of the function $2 - x$. For what values of x are the functions $\dfrac{1}{x^2}$ and $2 - x$ equal ? Hence solve the equation $x^3 - 2x^2 + 1 = 0$.

(ii) Draw the graph of the function $x - 2$ with the same axes, and solve the equation $\dfrac{1}{x^2} = x - 2$.

(iii) Draw the graph of the function $2x + 1$, and solve the equation $x^2(2x + 1) = 1$.

6. Draw graphs of the functions $\dfrac{1}{x}$ and $x^2 - 4$ with the same axes and to the same scale for values of x from -2 to $-0{\cdot}2$ and $0{\cdot}2$ to 3.

For what values of x are these functions equal ?

Hence solve the equation $x^3 - 4x = 1$.

B. REPRESENTATION OF FUNCTIONS.

If the area of a rectangle is fixed, the lengths of its sides cannot be chosen independently of one another ; the choice of a length for one side depends upon the length chosen for

the other. In other words, "the length of one side is a *known function* of the length of the other, when the area is given."

For example, if the area is 36 sq. inches, and if one side is of length x inches, then the other side must be of length $\frac{36}{x}$ inches.

Example II. Sketch the graph of the function $\frac{36}{x}$.

An accurate drawing is not required and it is not necessary to make a table of values. All that is necessary is to note a few important features of the function as follows :

(i) If x is positive, $\frac{36}{x}$ is positive ;
if x is negative, $\frac{36}{x}$ is negative.

(ii) If x is large and positive, $\frac{36}{x}$ is small, and by making x sufficiently large, we can make $\frac{36}{x}$ as small as we please.

In symbols, when $x \to \infty$, $\frac{36}{x} \to 0$, which reads in words "when x tends to or approaches infinity, $\frac{36}{x}$ tends to or approaches zero."

(iii) If x is small and positive, $\frac{36}{x}$ is large and positive $\left[e.g. \text{ if } x=0{\cdot}1, \frac{36}{x}=360\right]$.
In symbols, when $x \to +0$, $\frac{36}{x} \to +\infty$.

(iv) When x actually $=0$, $\frac{36}{x}$ has no meaning.

(v) For a negative value of x, the value of $\frac{36}{x}$ is equal in magnitude but opposite in sign to its value for the corresponding positive value of x.

In particular, when $x \to -0$, $\frac{36}{x} \to -\infty$,

and when $x \to -\infty$, $\frac{36}{x} \to -0$.

The rough shape of the graph can now be sketched (see Fig. 3).

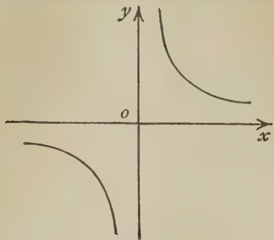

FIG. 3.

The graph is also said to correspond to the equation

$$y=\frac{36}{x} \quad \text{or} \quad xy=36.$$

EXERCISE I. b.

[In the following examples, squared paper should not be used.]

1. Explain in words how the value of the function $(x-1)(x-3)$ varies as x varies from 1 to 3.

For what values of x does the function $=0$?

For what range of values of x is the function positive ?

2. Can you find a value of x for which the function x^2-4x+4 is negative ?

Construct another function of x which has a similar property.

3. For what values of x is the function x^2+x-6 zero ?

For what range of values of x is the function x^2+x-6 negative ?

4. (i) What is the greatest value of the function $9-(x-1)^2$?

(ii) For what values of x is this function zero ?

(iii) To what value does this function tend when $x \to \infty$, and when $x \to -\infty$?

(iv) Sketch the graph of this function.

5. Sketch the graphs of (i) x^2, (ii) $(x-3)^2$, (iii) $(x+2)^2$.

6. Sketch the graphs of

(i) $+\sqrt{x}$, (ii) $-\sqrt{x}$, (iii) $\pm\sqrt{x-2}$, (iv) $\pm\sqrt{x+1}$.

7. (i) Can you find (*a*) a positive value of x, (*b*) a negative value of x, such that $\frac{1}{x^2} < 0{\cdot}01$?

(ii) When is this function $\frac{1}{x^2} > 100$?

(iii) Sketch the graph of $\frac{1}{x^2}$.

8. What is the value of the function $\frac{1}{x-3}$

(i) when x is large and positive, *e.g.* $+1003$?

(ii) when x is large and negative, *e.g.* -997 ?

(iii) when x is nearly equal to 3, (*a*) if $x>3$, *e.g.* $3{\cdot}001$, (*b*) if $x<3$, *e.g.* $2{\cdot}999$?

(iv) Sketch the graph of $\frac{1}{x-3}$.

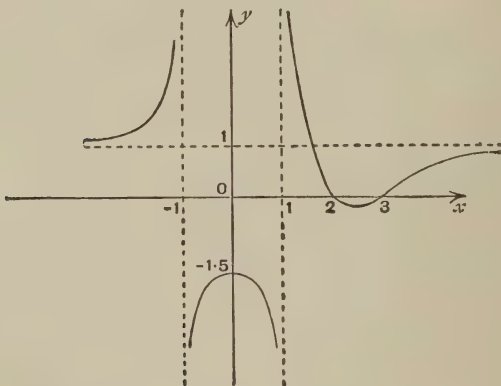

FIG. 4.

9. The graph of a certain function of x is shown in Fig. 4. Describe in words the variation in value of this function of x as x varies from $-\infty$ to $+\infty$.

10. ONP is a variable triangle with a right angle at N; ON lies along the axis Ox; the length of the hypotenuse OP is 5 and $ON=x$. What is the length of NP ?

As x varies, what is the locus of P ?

Of what function of x is this locus the graph ?

11. The graph of a certain function of x is shown in Fig. 5. Describe in words how the value of the function varies as x varies from -2 to $+4$. If the same series of values of the function recurs again and again every time x is increased by 7, what can you say about the value of the function as $x \to \infty$?

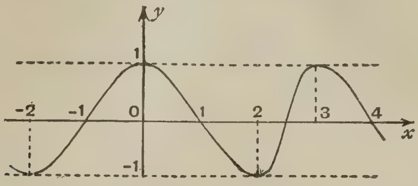

FIG. 5.

12. The values of a certain function of x can be calculated for all values of x, positive and negative. The function never has values greater than 1 or less than -2, and is zero when x equals -1 or 3 or 4. The function is positive if $-1 < x < 3$ or if $4 < x < \infty$, and for other values of x is negative.

Sketch the simplest graph of this function.

13. Sketch the simplest graph of a function which has the following characteristics :

(i) $\to +1$, when $x \to +\infty$.

(ii) $\to +\infty$, when $x \to +1$, provided $x > 1$.

(iii) Is never equal to 1.

(iv) Is not defined if $-1 < x < +1$.

(v) $\to -1$, when $x \to -\infty$.

(vi) $\to -\infty$, when $x \to -1$, provided $x < -1$.

14. (i) Can you find a value of x for which $\dfrac{x-1}{x-2}$ is firstly > 100 and secondly < -100 ?

(ii) For what value of x is this function zero ?

(iii) For what ranges of values of x is this function negative ?

(iv) Can you find a value of x for which the function equals 1 ?

(v) For what value of x is the function equal to 1·001 ?

(vi) For what value of x is the function equal to 0·999 ?

(vii) Sketch the graph of this function.

15. (i) For what values of x is the function $(x-1)(x-3)(x-4)$ zero ?

(ii) For what ranges of values of x is this function positive ?

(iii) Find the values of the function when $x=10, 0, -10$.

(iv) Can you find a value of x for which the function is $> 1{,}000{,}000$?

(v) Sketch the graph of this function.

16. (i) Can you find a value of x between 0 and 10 for which the function

$$\frac{(x-1)(x-5)}{x-3}$$

is firstly > 1000 and secondly < -1000 ?

(ii) Can you find the value of the function when $x=3$?

(iii) Describe the changes of sign in the function as x increases from -1 to $+8$, stating also where it vanishes.

(iv) What is the approximate error per cent. in taking the function as equal to x when $x=1000$ and when $x=-1000$?

(v) Sketch the graph of this function.

17. Sketch the graph of

(i) $(x-1)(x-2)(x-3)(x-4)(x-5)$;

(ii) $(x-1)(x-2)^2(x-4)(x-5)$;

(iii) $(x-1)(x-2)^3(x-5)$.

18. (i) Find approximately the value of the function

$$\frac{x-1}{(x-2)(x-3)}$$

if $x=1{\cdot}99,\ 2{\cdot}01,\ 2\tfrac{1}{2},\ 2{\cdot}99,\ 3{\cdot}01$.

(ii) What can you say about the value of this function when x is large ; about how much, for example, is it if x is a million ?

(iii) What is its approximate value if $x=-1{,}000{,}000$?

(iv) For what range of values of x is this function negative ?

(v) Sketch the graph of the function.

19. Sketch the graph of $\dfrac{x-2}{(x-1)(x-3)}$.

20. Sketch the graph of $\dfrac{x-1}{(x-2)^2}$.

21. Sketch the graph of $\dfrac{(x+1)(x-6)}{(x-1)(x-4)}$.

22. Sketch the graph of $\dfrac{x(x-3)}{(x+2)(x-1)}$.

C. CONSTRUCTION OF FUNCTIONS.

Example III. AB is a variable chord of a circle, centre O, of radius 5 cm.; $AB=2x$ cm., and the height CD of the

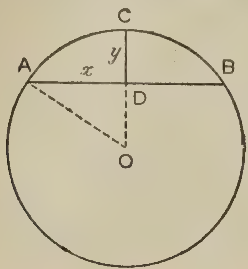

FIG. 6.

minor segment cut off by AB is y cm. Express x as a function of y.

$$AO=5 \text{ cm.}, \quad OD=OC-DC=5-y \text{ cm.}$$

By Pythagoras, $$x^2+(5-y)^2=5^2;$$

$$\therefore\ x^2+25-10y+y^2=25 \quad \text{or} \quad x^2=10y-y^2=y(10-y);$$

$$\therefore\ x=\sqrt{\{y(10-y)\}}.$$

Coordinates of a Point. The position of a point in a plane is often fixed by giving its distances from two perpendicular straight lines Ox, Oy (see Fig. 7). If PN is perpendicular to Ox, the length of ON equals the distance of P from Oy, and is called the *x-coordinate* or the *abscissa* of P, and the length of PN, which is the distance of P from Ox, is called the *y-coordinate* or the *ordinate* of P.

With the data of Exercise I. c, No. 1, the statement that the coordinates of A are $(2, 3)$ means that $OH=2$, $HA=3$; the x-coordinate is always put first. And if $ON=x$, $NP=y$, we say that the coordinates of P are (x, y). We may also express the fact as follows: if you start from the origin O and move x units x-wards and then y units y-wards, you arrive at P.

EXERCISE I. c.

1. In Fig. 7 the coordinates of A, B, P are respectively (2, 3); (6, 5); (x, y). Express y as a function of x.

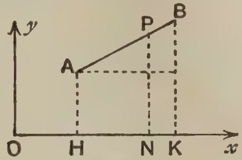

FIG. 7.

2. AB is the diameter of a circle APB; PN is the perpendicular from P to AB; if $AB=8$, $AN=x$, $PN=y$, express y as a function of x.

3. $ABCD$ is a rectangle; P is a point such that the perpendicular PN from P to AB is equal to PC. If $BC=4$, $PN=x$, $NB=y$, express y as a function of x.

4. PN is an altitude of the triangle APB; O is the mid-point of AB. If $PN=y$, $ON=x$, $AB=6$ and $PA^2+PB^2=30$, express y as a function of x.

5. AB is a diameter of a circle; a line AQR cuts the circle at Q and the tangent at B in R; P is a point on AQ such that $AP=QR$; PN, QK are the perpendiculars from P, Q to AB; $AB=4$, $AN=x$, $PN=y$, $QK=z$; express (i) z as a function of x (use the fact that $BK=AN$), (ii) y as a function of x.

Sketch the graph of y.

6. AOB is a triangle right-angled at O; PN is the perpendicular from a point P on AB to OB; if $AP=3$, $PB=4$, $ON=x$, $PN=y$, express y as a function of x.

7. A point R is taken on the side AB of a triangle ABC of area z sq. inches such that $AR=x \,.\, AB$, where $x>\frac{1}{2}$. RQ, RH are drawn parallel to BC, AC to meet AC, BC at Q, H; QK is drawn parallel to AB to meet BC at K. Express the area of $QRHK$ as a function of x and z. (C.S.C.)

8. AB is a diameter of a circle; CD is a chord parallel to AB and at distance b inches from it; any chord AQ cuts CD at R; RN is drawn perpendicular to AB; QP is drawn parallel to AB and cutting RN at P; if $AB=a$, $AN=x$, $NP=y$ inches, express y as a function of x.

9. AB is a fixed diameter of a given circle; the tangent at B meets a variable chord AP at Q; $AB=d$, $AP=x$, $BQ=y$; express y as a function of x.

10. $ABCD$ is a straight line and AH, BK, CL are three fixed lines perpendicular to it; $AB=a$, $BC=b$, $CD=c$; a variable line cuts AH, BK, CL at P, Q, R and DQ cuts CL at S; if $AP=y$, $CR=z$, $CS=x$, express x as a function of y, z.

11. Assuming the length, breadth and depth of an ordinary match-box are in the ratio 10 : 7 : 3, express the volume of the box as a function of the area of match-board used in making it, *i.e.* box and drawer.

12. Taking Ox, Oy as perpendicular axes, sketch the graphs of x^2 and $\frac{1}{x}$ for positive values of x; any line perpendicular to Ox cuts the graphs at P, Q and Ox at N; if $ON=x$, express the area of the triangle OPQ as a function of x.

D. FUNCTIONAL NOTATION.

Any expression whose value can be determined when the value of x is known, can be represented by the symbol $f(x)$.

$f(x)$ is shorthand for the words "a function of x."

In a particular question $f(x)$ might be used to represent the function $5x^4-2x+8$.

Then $f(2)$ would mean the value of this function when $x=2$.

$$\therefore\ f(2) \text{ would mean } 5(2^4)-2(2)+8 = 80-4+8 = 84,$$

and

$$f(-1) \text{ would mean } 5(-1)^4-2(-1)+8 = 5+2+8 = 15,$$

and

$$f(0) \text{ would mean } 5(0)^4-2(0)+8 = 8.$$

Example IV. If $f(x)\equiv x^2-3+\frac{1}{x}$, find the value of $f(5)$ and $f(2a)$.

$$f(5)=5^2-3+\tfrac{1}{5}=25-3+\tfrac{1}{5}=22\tfrac{1}{5},$$

$$f(2a)=(2a)^2-3+\frac{1}{2a}=4a^2-3+\frac{1}{2a}.$$

Example V. If $f(x) \equiv x^2 + 2x$, find the value of $f(x+h) - f(x)$.

Here
$$f(x+h) = (x+h)^2 + 2(x+h)$$
$$= x^2 + 2xh + h^2 + 2x + 2h\ ;$$
$$\therefore\ f(x+h) - f(x) = x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x$$
$$= 2xh + h^2 + 2h$$
$$= \mathbf{h(2x+2+h)}.$$

EXERCISE I. d.

1. If $f(x) \equiv x^2 + 2$, find the values of $f(1)$; $f(0)$; $f(-1)$; $f(2a)$; $f(b^3)$.

2. If $f(x) \equiv 10^x$, find the values of $f(1)$; $f(2)$; $f(0)$; $f(-1)$; $f(2x)$.

3. If $f(x) \equiv \log x$, find the values of $f(1000)$; $f(2)$; $f(20)$; $f(1)$; $f(x^3)$.

4. If $f(x) \equiv x^2 - 3x + 5$, find the values of $f(2)$; $f(x+h)$; $f\left(\frac{1}{x}\right)$.

5. If $f(x) \equiv x^2 + 3x$, find the value of $\dfrac{f(x+h) - f(x)}{h}$. What is the approximate value of this when h is small compared with x ?

6. If $f(x) \equiv \dfrac{1}{x}$, find the values of $f(1)$ and $\dfrac{f(1+h) - f(1)}{h}$. What is the approximate value of this last expression when h is small ?

7. If $f(x) \equiv x^2 + 5$, simplify
   (i) $f(3x)$; (ii) $f(x+1) + f(x-1) - 2f(x)$.

8. If $f(x) \equiv x^2$, simplify
   (i) $f(x^3)$; (ii) $f(x+h) - f(x) - \{f(x) - f(x-h)\}$.

E. EQUATION OF A LINE OR CURVE.

Figure 8 represents the graph of the function $\frac{x}{2} + 1$. We know that this is a straight line.

If, from a point P on the line, we draw PN perpendicular to Ox, then ON, NP are the coordinates of P. If, for example, $ON = 3$, we know that $PN = \frac{3}{2} + 1 = 2\frac{1}{2}$. If $ON = x$ and $NP = y$, we know that $y = \dfrac{x}{2} + 1$, and this relation is called the *equation of the line*, because it connects the coordinates of each separate point on the line.

B is a point on the line whose x-distance is zero; and when $x=0$, $y=\frac{x}{2}+1=0+1=1$; $\therefore OB=1$.

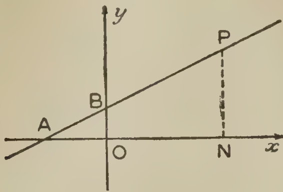

FIG. 8.

A is a point on the line whose y-distance is zero; and when $y=0$, we have $0=\frac{x}{2}+1$; $\therefore \frac{x}{2}=-1$ or $x=-2$.

$\therefore OA=-2$, *i.e.* A is 2 units of length to the *left* of O.

It is easy to find whether any given point lies on the line or not, by simple substitution in its equation: thus the point $(-6, -2)$ lies on the line, because $-2=-\frac{6}{2}+1$.

Fig. 2 on p. 2 represents part of the graph of the function $x^2-8x+32$. We therefore say that the *equation of this curve* is $y=x^2-8x+32$, because the x-distance and the y-distance of each point on the curve are connected by the relation

$$y=x^2-8x+32.$$

EXERCISE I. e.

1. Fig. 9 represents the line whose equation is $y=\frac{2x}{3}+2$.

(i) If $ON=6$, what is PN?
(ii) If $PN=4$, what is ON?
(iii) If $OK=-4$, what is KS?
(iv) If $KS=-1$, what is OK?
(v) What are OA, OB?
(vi) If $ON=3c$, what is PN?
(vii) Do $(30, 22)$, $(-12, -10)$ lie on the line?

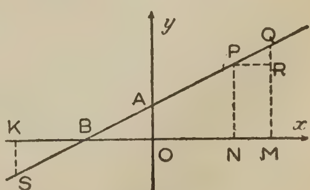

FIG. 9.

2. Fig. 10 represents the line whose equation is $y=3-\dfrac{3x}{5}$.

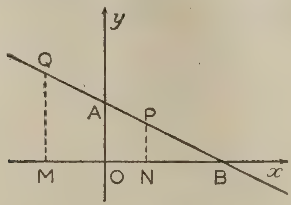

FIG. 10.

(i) If $ON=4$, what is PN ? (ii) If $PN=1$, what is ON ?
(iii) If $OM=-2$, what is QM ? (iv) If $QM=6$, what is OM ?
(v) What are OA, OB ? (vi) If $ON=2a$, what is PN ?
(vii) Do $(20, -9)$; $(2\frac{1}{2}, 1\frac{1}{2})$; $(-30, 24)$ lie on the line ?

3. With the data of No. 1, (i) if $ON=6$, $NM=1$, calculate PN, QM, $\dfrac{QR}{PR}$; (ii) if $ON=a$, $NM=h$, calculate PN, QM, $\dfrac{QR}{PR}$, where R is the foot of the perpendicular from P to QM.

4. If Fig. 16, p. 22, represents the curve whose equation is $y=\frac{1}{2}x^2$, calculate $\dfrac{QR}{PR}$ (i) if $ON=1$, $NM=\frac{1}{2}$, (ii) if $ON=1$, $MN=0{\cdot}1$, (iii) if $ON=a$, $NM=h$.

5. The equation of the curve in Fig. 11 is $y=(x+1)(5-x)$.

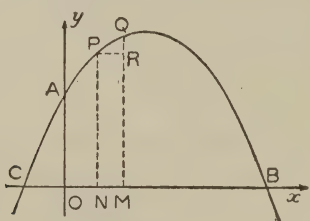

FIG. 11.

(i) Find OA, OB, OC.
(ii) Find PN if $ON=1$ and if $ON=6$, and interpret the result.
(iii) Find ON if $PN=8$ and if $PN=-7$ and if $PN=9$, and interpret your answers.
(iv) Find $\dfrac{QR}{PR}$ if $ON=1$, $NM=0{\cdot}1$ and if $ON=1$, $NM=h$, and if $ON=a$, $NM=h$.

6. Sketch lines or curves corresponding to the following equations :

(i) $y=2x-1$; (ii) $x+y=1$; (iii) $y=3x^2+1$;
(iv) $y=(x-1)(x-6)$; (v) $x^2+y^2=25$.

7. PN, QM are the ordinates of two points P, Q on the curve $xy=36$ (Fig. 3, p. 7) ; PR is drawn perpendicular to QM.

(i) If $ON=3$, $NM=1$, calculate PN, QM, QR.

(ii) If $ON=a$, $NM=h$, calculate QR and $\dfrac{QR}{PR}$.

8. Fig. 12 is a curve whose equation is written $y=f(x)$; draw lines in the figure whose lengths represent $f(4)$, $f(2)$, $f(-1)$, $f(0)$.

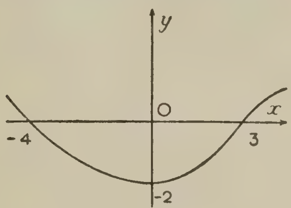

FIG. 12.

What do you know about the values of $f(3)$, $f(0)$, $f(-4)$?
Which of the following are positive : $f(5)$, $f(1)$, $f(-2)$, $f(-5)$?

9. If the equation of the curve in Fig. 11 is $y=f(x)$ and if $ON=a$, $NM=h$, write down expressions for PN, QM, $\dfrac{QR}{PR}$.

CHAPTER II.

LIMITS AND GRADIENTS.

It has already been noticed, in connection with the graphical representation of functions, that a function may have no meaning for one or more special values of the variable (*e.g.* $\dfrac{36}{x}$ has no meaning when $x=0$, see p. 6). We shall now examine more closely the behaviour of a function in the neighbourhood of such values. Examples I. and II. illustrate the meaning of a "Limit" of a function.

Example I. Plot the values of the functions:

$$\text{(i)}\ 1-\frac{1}{n}; \qquad \text{(ii)}\ 1+\frac{1}{n}; \qquad \text{(iii)}\ 1+(-1)^n\cdot\frac{1}{n}$$

for positive *integral* values of n.

(i) $1-\dfrac{1}{n}$.

We have the following table of values:

$n=1$	2	3	4	5	6	7	8
$1-\dfrac{1}{n}=0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

which are represented in Fig. 13.

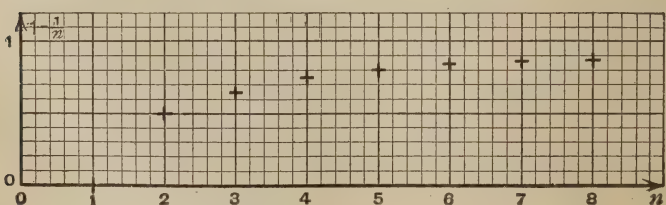

Fig. 13.

(ii) $1+\frac{1}{n}$.

We have the following table of values :

$n=1$	2	3	4	5	6	7	8
$1+\frac{1}{n}=2$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$

which are represented in Fig. 14.

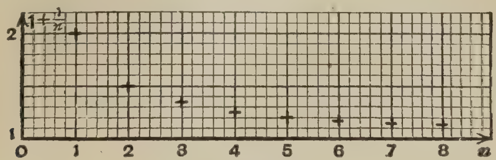

FIG. 14.

(iii) $1+(-1)^n \cdot \frac{1}{n}$.

We have the following table of values :

$n=$ 1	2	3	4	5	6	7	8
$1+(-1)^n \cdot \frac{1}{n}=0$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{5}{4}$	$\frac{4}{5}$	$\frac{7}{6}$	$\frac{6}{7}$	$\frac{9}{8}$

which are represented in Fig. 15.

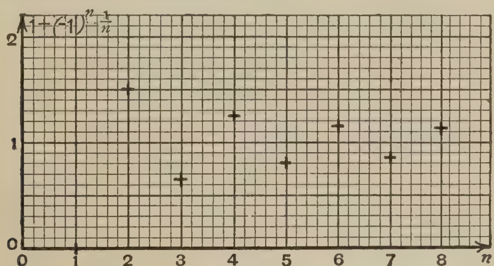

FIG. 15.

Either from the graphs or from examining the functions direct we note that

(i) $1-\frac{1}{n}$ is always less than 1, and that, as n increases, the function steadily increases and approaches the value 1.

(ii) $1+\frac{1}{n}$ is always greater than 1, and that as n increases the function steadily decreases and approaches the value 1.

(iii) $1+(-1)^n\cdot\frac{1}{n}$ is alternately less and greater than 1, and that as n increases the function increases and decreases alternately, but approaches the value 1.

In each case it is possible to find a value of n for which and for all greater values the function differs from 1 by less than a given amount, however small.

E.g. each function differs from 1 by less than 0·001 if $n > 1000$.

But it is impossible to find a value of n for which any of the functions actually equals 1.

Under these conditions we say that:

The Limit of $1+\frac{1}{n}$ as n tends to infinity is 1 [but the limit is not attained in this case]. And we write it as follows:

$$\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)=1.$$

Similarly $\underset{n\to\infty}{\text{Lt}}\left(1-\frac{1}{n}\right)=1$ and $\underset{n\to\infty}{\text{Lt}}\left[1+(-1)^n\cdot\frac{1}{n}\right]=1.$

Example II. What is the limit of $\frac{(1+h)^2-1}{h}$ as h tends to 0?

$$\frac{(1+h)^2-1}{h}=\frac{1+2h+h^2-1}{h}=\frac{2h+h^2}{h}.$$

If $h=0$, $\frac{2h+h^2}{h}$ becomes $\frac{0}{0}$, which is a meaningless expression.

If $h\neq 0$, $\frac{2h+h^2}{h}=2+h.$

The smaller h becomes, the closer $2+h$ approaches to 2. It never attains the value 2, for h is never actually 0. But we can choose h so that $2+h$ differs from 2 for that and all smaller values of h by less than any given amount, however small;

$\therefore$ its limit is 2;

$$\therefore \underset{h\to 0}{\text{Lt}}\frac{(1+h)^2-1}{h}=2 \text{ (limit not attained).}$$

EXERCISE II. a.

1. (i) Plot the values of $\frac{2n+1}{n+1}$ for integral values of n from 3 to 10.

(ii) Can you find a value of n for which $\frac{2n+1}{n+1}$ differs from 2 by less than 0·001 ?

(iii) Can you find a value of n for which $\frac{2n+1}{n+1}$ equals 2 ?

(iv) What is the limit of $\frac{2n+1}{n+1}$ as n tends to infinity ?

2. (i) If $s_1=1$; $s_2=1+\frac{1}{2}$; $s_3=1+\frac{1}{2}+\frac{1}{4}$; $s_4=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, etc., plot the values of s_n for values of n from 1 to 5.

(ii) Can you find a value of n for which s_n differs from 2 by less than 0·001 ?

(iii) Can you find a value of n for which s_n equals 2 ?

(iv) What is the value of $\underset{n\to\infty}{\text{Lt}}\, s_n$?

3. (i) Express as a decimal to 3 figures the value of $\frac{n^2}{2n^2+1}$ when $n=5$ and $n=10$.

(ii) What is the limit of $\frac{n^2}{2n^2+1}$ when n tends to infinity ?

(iii) Is this limit attained ?

4. (i) Find the values of $\frac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n\to\infty}{\text{Lt}}\, \frac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$?

5. (i) Plot the values of $\frac{2^n}{1+2^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n\to\infty}{\text{Lt}}\, \frac{2^n}{1+2^n}$?

(iii) For what integral value of n does $\frac{2^n}{1+2^n}$ differ from 1 by less than $\frac{1}{100}$?

6. (i) Find the values of $\dfrac{x^2-1}{x^2-x}$ for $x=2$, $1\cdot5$, $1\cdot1$, $1\cdot01$.

(ii) What is the value of $\underset{x\to1}{\text{Lt}}\ \dfrac{x^2-1}{x^2-x}$?

(iii) Is this limit attained?

(iv) Find a value of x for which $\dfrac{x^2-1}{x^2-x}$ differs from 2 by less than $0\cdot001$.

7. Fig. 16 shows part of the graph of $y=\frac{1}{2}x^3$.

$$ON=1, \quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\dfrac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\dfrac{QR}{PR}$ as $h\to0$? Interpret this result geometrically.

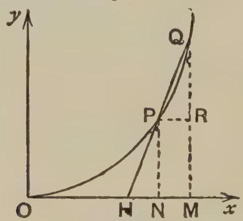

FIG. 16.

8. If a stone is dropped in a vacuum, it falls s feet in t seconds, where $s=16t^2$; the graph of this is represented in Fig. 17.

$$ON=\tfrac{1}{2}, \quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\dfrac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\dfrac{QR}{PR}$ as $h\to0$? Interpret this result.

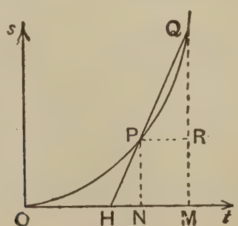

FIG. 17.

9. (i) Draw the graph of $5x-x^2$ from $x=0$ to 5.

(ii) Draw the ordinate corresponding to $x=1{\cdot}5$. What is its length ?

(iii) Take any point A on the x-axis and call $OA=a$. Give a geometrical meaning to the ratio

$$\frac{[5(a+h)-(a+h)^2]-[5a-a^2]}{h}.$$

(iv) What is the limit of this function when $h\to 0$?

(v) For what value of a is this limit equal to 0 ? Interpret this result geometrically.

10. (i) Draw the graph of $\frac{1}{x}$ from $x=4$ to $\frac{1}{4}$.

(ii) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}$.

(iii) What is the limit of this ratio when $h\to 0$? Interpret this result.

(iv) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$; evaluate its limit when $h\to 0$; and find the value of a for which this limit equals $-\frac{1}{4}$.

11. (i) What is the average of the numbers $1, 2, 3, 4, \ldots, (n-1)$?

(ii) Prove that their sum is $\frac{n(n-1)}{2}$.

(iii) Find the value of $\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n^2}[1+2+3+\ldots+(n-1)]$.

12. ABC is an isosceles right-angled triangle; $BC=10$ cm.; BC is divided into n equal parts, and through each point of division a line is drawn parallel to BA to meet CA, and rectangles are completed as in the figure. By using Ex. 11,

(i) find the sum of the areas of all the rectangles in terms of n;

(ii) find the limit of this sum when n tends to infinity;

(iii) what is the area of the triangle ABC ?

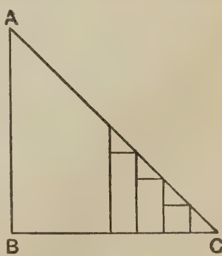

FIG. 18.

13. It can be proved that the sum of the series

$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

(i) Find the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n^3}(1^2+2^2+3^2+\ldots+n^2)$.

(ii) Fig. 19 gives the graph of $y=x^2$; $OA=1$; OA is divided into n equal parts, and through each point of division lines are drawn parallel to Oy, and rectangles are completed as in the figure. Express the sum of the areas of all these rectangles in terms of n, and find the limit of this sum when n tends to infinity.

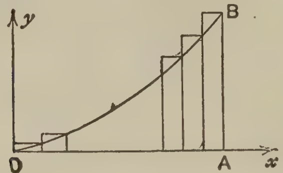

FIG. 19.

14. (i) Use the result given in No. 13 to find the value of

$$\frac{1}{n^3}[1^2+2^2+3^2+\ldots+(n-1)^2].$$

(ii) If the rectangles are drawn so that each ends below the graph of $y=x^2$ (the arrangement in Fig. 18), find the sum of the areas of the rectangles in terms of n; and find the limit of this sum when $n\to\infty$.

(iii) By comparing this with the result in Ex. 13, what can you say about the area of the figure bounded by OA, AB and the curve OB in Fig. 19?

CALCULATION OF RATES OF CHANGE FROM STATISTICS.

Example III. The following table gives the height of the mercury barometer at intervals of two hours during a day:

Time	8 a.m.	10 a.m.	Noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Height in inches	28·57	28·65	28·85	29·10	29·22	29·12	29·0

Find the average rate at which the barometer was rising

(i) between 10 a.m. and noon ;

(ii) between noon and 2 p.m. ;

(iii) between 4 p.m. and 6 p.m.

Draw the barograph for the day, and find the rate at which the barometer was probably rising at noon.

(i) Between 10 a.m. and noon the barometer rose $0{\cdot}2''$ in 2 hours;

$\therefore$ the average rate of rise was $0{\cdot}1''$ per hour.

(ii) Between noon and 2 p.m. the barometer rose $0{\cdot}25''$ in 2 hours ;

$\therefore$ the average rate of rise was $0{\cdot}125''$ per hour.

(iii) Between 4 p.m. and 6 p.m. the barometer fell $0{\cdot}1''$ in 2 hours ;

$\therefore$ the average rate of rise was $-0{\cdot}05''$ per hour.

Figure 20 gives the required barograph.

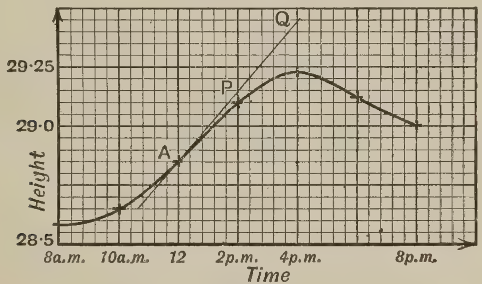

FIG. 20.

It is a curve and not a straight line, because the rate is altering throughout the day. If it continued to rise at the same rate after 12 o'clock as it is rising at 12 o'clock, the barograph would be the straight line formed by drawing the tangent APQ at the point A, which corresponds to 12 o'clock on the graph. In the figure, this tangent is drawn by eye, and cuts the 2 o'clock and 4 o'clock lines (or any other convenient lines) at P and Q. Now the difference in heights registered by P, Q is $0{\cdot}30''$.

$\therefore$ the rate of rise at A is $0{\cdot}30''$ in 2 hours,

or $0{\cdot}15''$ per hour.

EXERCISE II. b.

1. The following table gives the population of the United Kingdom for various years :

Year	1840	1850	1860	1870	1880	1890	1900
Population (in millions)	27·0	27·7	29·3	31·8	35·2	38·1	42·0

[Give answers in thousands per year.]

(*a*) What was the average rate of increase of population in the period 1840 to 1850 ?

(*b*) What was the average rate of increase of population for the whole period 1840 to 1900 ?

(*c*) During what period of ten years was the population increasing most rapidly, and what was the rate per year then ?

2. At the end of a minutes a car has travelled x miles, and at the end of b minutes it has travelled y miles. What was its average speed

(*a*) for the first a minutes ?

(*b*) for the first b minutes ?

(*c*) during the period a minutes to b minutes ?

If in the above question $a=3$ and $b=4$, what was the average speed of the car during the 4th minute ?

3. The following table gives the distances a car travels, starting from rest :

Time in minutes	5	10	15	20	25	30	35	40	45
Distance in miles	0·7	2·6	5·2	8·3	11·4	14·2	16·7	19	20·5

[Give answers in the form miles per hour.]

(*a*) What was the average speed of the car for the first 5 minutes ?

(*b*) What was the average speed for the first 20 minutes ?

(*c*) What was the average speed for the whole period, 45 minutes ?

(*d*) Draw a graph to illustrate the motion of the car, and by drawing a tangent to the graph at the point determined by $t=35$, find the probable speed of the car 35 minutes after it started.

(*e*) By drawing a tangent to the graph where it appears to be steepest, find the greatest speed attained by the car.

4. A lift ascends 90 ft. in 40 seconds, and its height h feet at intervals of 5 seconds is given by the following table :

t	5	10	15	20	25	30	35	40
h	4	15	30	45	62	77	86	90

Draw a graph to illustrate the motion, and draw tangents to the graph at the points determined by $t=10$, $t=20$, $t=30$.

(i) What is the average speed of the lift for the first 10 seconds ?

(ii) Estimate from your graph the speed of the lift after 10 seconds ?

(iii) What is the average speed of the lift for the first 20 seconds ?

(iv) Estimate its speed after 20 seconds.

(v) What is the average speed of the lift for the last 10 seconds ?

(vi) Estimate its speed after 30 seconds.

5. The weight that can be carried by a certain type of bridge varies with the diameters of the spars used in accordance with the following table :

Diameter of spar in inches -	3	5	7	9	11
Load carried in tons - -	0·1	0·7	2·4	6	12

[Give answers in the form tons per inch.]

What is the average rate at which the load carried increases when the diameter is increased from

(*a*) 3 inches to 5 inches ?
(*b*) 7 inches to 9 inches ?
(*c*) 9 inches to 11 inches ?
(*d*) 7 inches to 11 inches ?

Estimate the 'rate' of increase of the load when the diameter is 7 inches by drawing a graph.

6. Water runs out of a bath, and the volume that runs out is given by the following table :

Time in seconds - -	5	10	15	20	25
Volume in cubic feet - -	6	9	10·5	11·2	11·6

Draw a graph and find the rate of flow at the end of each period of 5 seconds. Show from your results that the rate of flow is proportional to the volume of water remaining in the bath, there being 12 cubic feet of water in the bath originally.

SUMMARY OF RESULTS.

(i) It is possible to have a function of x to which, for one or more special values of x, no meaning can be given.

E.g. $\dfrac{x^2-1}{x-1}$ has no meaning when $x=1$.

(ii) In such cases, the function may tend towards a definite limit as x tends to that value; but the limit is not attained.

E.g. $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-1}{x-1}=2$, but no value of x exists for which $\dfrac{x^2-1}{x-1}$ equals 2.

(iii) If a function of x is represented by a graph, the rate at which the function is increasing for any value of x is represented by the slope of the tangent to the graph at the point corresponding to that value of x, provided that the units used in drawing the graph are taken into account.

CALCULATION OF RATES OF CHANGE FROM FORMULAE.

Example IV. Draw the graph of $y=2+3x+x^2$, and find the change of y per unit increase of x, when (i) $x=2$, (ii) $x=a$.

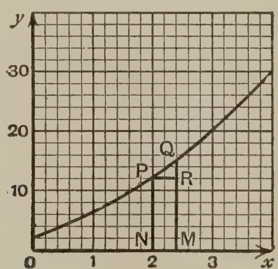

FIG. 21.

We have, by calculation, the table:

$x=0$	1	2	3	4
$y=2$	6	12	20	30

which is represented in Figure 21.

(i) When $ON=2$, $PN=2+3\times 2+2^2=2+6+4=12$.
When $OM=2+h$, $QM=2+3(2+h)+(2+h)^2=12+7h+h^2$;

$\therefore$ when x increases by h, y increases by $QM-PN$ or $QR=7h+h^2$;

$\therefore$ the average increase of y per unit increase of $x=\dfrac{7h+h^2}{h}=7+h$

or
$$\frac{QR}{PR}=7+h;$$
$$\therefore \operatorname*{Lt}_{h\to 0}\frac{QR}{PR}=7;$$

$\therefore$ at $x=2$, the rate of increase of y with respect to x is 7.

(ii) When $ON=a$, $PN=2+3a+a^2$.
When $OM=a+h$,
$$QM=2+3(a+h)+(a+h)^2=2+3a+3h+a^2+2ah+h^2;$$
$$\therefore QR=QM-PN=3h+2ah+h^2;$$
$$\therefore \frac{QR}{PR}=\frac{3h+2ah+h^2}{h}=3+2a+h;$$

$\therefore$ at $x=a$, the rate of increase of y with respect to x,
$$=\operatorname*{Lt}_{h\to 0}(3+2a+h)$$
$$=3+2a.$$

Definition. In Figure 21, $\dfrac{QR}{PR}$ is called the *average gradient* of the graph over the interval NM.

The limit of $\dfrac{QR}{PR}$ when h tends to 0 or $\operatorname*{Lt}_{h\to 0}\dfrac{QR}{PR}$ is called the *gradient* of the graph at P, where $PR=NM=h$.

Figure 22 illustrates the way in which the "average gradient" changes as the point Q is taken nearer and nearer to P.

Q_1, Q_2, Q_3 are successive positions of Q, whilst P remains fixed.

The average gradient over the interval MN is $\dfrac{QR}{PR}$.

The limiting position to which the chord PQ tends as arc $PQ \to 0$ is indicated by the tangent TPT'.

$$\mathop{\mathrm{Lt}}_{h\to 0} \frac{QR}{PR} = \text{gradient at } P = \text{gradient of the tangent } PT = \frac{PM}{MT}.$$

The gradient measures the "rate of change of y with respect to x," or the change in y per unit increase in x.

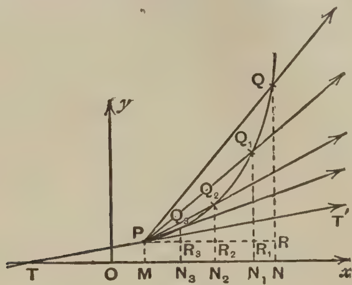

FIG. 22.

EXERCISE II. c.

1. What is the gradient of the slopes shown in Figs. 23-25?

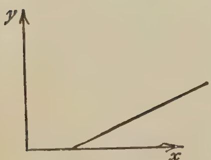

FIG. 23.

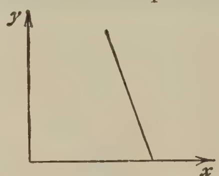

FIG. 24.

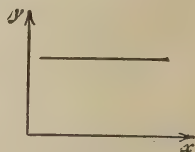

FIG. 25.

2. Figure 26 represents a hill; horizontal scale x-axis is 1 inch : 100 yards; vertical scale y-axis is 1 inch : 10 feet. Find (i) the average gradient from A to B, (ii) the average gradient from P to Q, (iii) the gradients at C, D, Q.

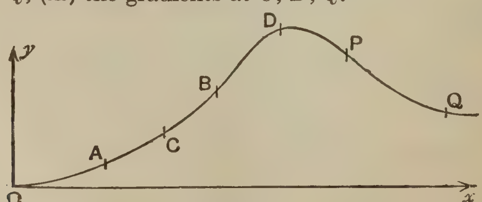

FIG. 26.

3. Draw lines of gradients (i) $\frac{3}{4}$, (ii) $-\frac{1}{2}$.

4. Find the coordinates of a point on the curve in Fig. 26, for the scale given in Ex. 2, where the slope is $\frac{1}{40}$.

5. (i) Draw the graph of $y=2x+3$.

(ii) Show that the points (1, 5) and (4, 11) lie on it.

(iii) What is the average gradient of the graph between these two points ?

(iv) What is the average gradient of the graph over the interval $x=2$ to $x=6$?

(v) What is the value of y when $x=a$ and when $x=a+h$, and what is the average gradient of the graph over this interval ?

(vi) Why does the average gradient of the graph not depend on the values of either a or h ?

6. (i) Draw the graph of $y=7-5x$.

(ii) What is the average gradient of the graph over the interval $x=3$ to $x=7$?

(iii) What is the gradient of the graph when $x=a$?

7. What is the gradient of the straight line joining the points (2, 5) and (7, 8) ?

8. A marble rolling down an inclined plane travels s feet in t seconds, where $s=3t^2$.

(i) How far has the marble travelled in 1 second ? What is its average speed for the 1st second ?

(ii) How far has the marble travelled in 2 seconds ? What is its average speed for the 1st two seconds ? What is its average speed during the 2nd second ?

(iii) How far has the marble travelled in 2·1 seconds ? What is its average speed in the interval 2 seconds to 2·1 seconds ?

(iv) How far has it travelled in $(2+h)$ seconds ? What is its average speed in the h seconds between 2 seconds and $(2+h)$ seconds ?

(v) What does your answer to Question (iv) become when $h=0{\cdot}1$ second, when $h=0{\cdot}01$ second and when $h=0{\cdot}000001$ second ?

What is its speed exactly 2 seconds after it begins to move ?

Draw a graph to illustrate the motion from $t=0$ to $t=3$, and find its speed after 2 seconds by drawing a tangent.

9. The distance d ft. that a stone has fallen after t seconds is given by the formula $d = 16t^2$.

(i) How far has the stone fallen after 3 seconds? What is its average speed for the first 3 seconds?

(ii) How far has the stone fallen after 2 seconds? What is its average speed during the third second?

(iii) How far has the stone fallen after 2·9 seconds? What is its average speed during the interval from 2·9 seconds to 3 seconds?

(iv) How far does it fall in $(3-h)$ seconds? What is its average speed during the h seconds from $(3-h)$ seconds to 3 seconds?

(v) What does your answer to Question (iv) become when $h = 0{\cdot}1$ second, when $h = 0{\cdot}01$ second and when $h = 0{\cdot}000001$ second?

What is the velocity of the stone 3 seconds after it is dropped?

10. In the same way as in Ex. 9 work out the speed of the stone 1 second after it is dropped.

11. In the same way as in Ex. 9 work out the speed of the stone a seconds after it is dropped. Evaluate your result for $a = 1$, 2 and 3, and compare with previous results.

12. The distance in feet travelled in t seconds by a body moving in a straight line from a fixed point A is given by the formula $AP = 3t^2 + 5t + 1$. Find AP when $t = 2$ and when $t = 2 + h$. Find the average speed of the body P for the h seconds commencing with the end of the 2nd second. What is this speed when $h = 0{\cdot}1$ sec., when $h = 0{\cdot}01$ sec., when $h = 0{\cdot}0001$ sec. and when $h = -0{\cdot}0000001$ sec. What is the speed of P 2 seconds after the motion starts?

13. Draw a line AB of length 10 cm. and describe a semicircle with AB as diameter; P is any point on the semicircle and PN is the perpendicular to AB. Let $AN = x$ and $PN = y$. Find by measurement or calculation the average gradient of the semicircle over the intervals (i) $x = 0$ to 1; (ii) $x = 0$ to 2; (iii) $x = 3$ to 7; (iv) $x = 8$ to 10. Interpret your results geometrically.

14. The area of a circular blot of ink is increasing at a steady rate of 2 sq. cm. per sec.; if the radius is x cm. after t sec., find the average rate of increase of x over the interval (i) $t = 5$ to $t = 10$; (ii) $t = 5$ to $t = 5{\cdot}1$, assuming that $x = 0$ when $t = 0$.

15. The perimeter of a rectangle is 24 inches: if its area is y sq. in. when one side is of length x in., find

(i) the average rate of change of y over the interval $x=3$ to 5;

(ii) the average rate of change of y over the interval $x=3$ to $3+h$;

(iii) the average rate of change of y over the interval $x=a$ to $a+h$;

(iv) the rate of change of y when $x=a$;

(v) the value of x when the rate of change of y is zero; what does this mean?

(vi) Express y as a function of x and draw its graph.

16. If $y=3x+x^2$, find (i) the average rate of change of y over the interval $x=1$ to $1{\cdot}1$; (ii) the gradient at $x=1$; (iii) the value of x when the gradient is zero.

17. Calculate the gradient of the graph of $\frac{1}{x}$ when (i) $x=2$, (ii) $x=a$.

18. Calculate the gradient of the graph of $2x^3$ when $x=c$.

19. A marble rolling down a groove travels s feet in t seconds where $s=\frac{1}{4}t^2$; find (i) the average rate of change of s over the interval $t=1$ to $t=2$, (ii) the rate of change of s when $t=1$. What does this mean?

20. The graph of $y=mx+c$ is a straight line, m and c being any constant numbers. Find its gradient.

21. Find the gradient of $y=ax^2+bx+c$ when $x=2$.

22. Find the gradient of $y=ax^2+bx+c$ when $x=x_1$. For what value of x is this gradient equal to zero? What is the geometrical significance of a zero gradient?

23. Give geometrical meanings to the following, taking x to be the x-coordinate of a graph (do not simplify the expressions):

(i) $\dfrac{(x+h)^2-x^2}{h}$; (ii) $\underset{h\to 0}{\mathrm{Lt}}\dfrac{(x+h)^2-x^2}{h}$;

(iii) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$; (iv) $\underset{h\to 0}{\mathrm{Lt}}\dfrac{\sqrt{x-h}-\sqrt{x}}{h}$;

(v) $\underset{h\to 0}{\mathrm{Lt}}\dfrac{\pi(r+h)^2-\pi r^2}{h}$; (vi) $\dfrac{\frac{4}{3}\pi(r+h)^3-\frac{4}{3}\pi r^3}{h}$.

24. If $f(x)\equiv x^2$, evaluate $\underset{h\to 0}{\mathrm{Lt}}\dfrac{f(x+h)-f(x)}{h}$ and interpret the result.

25. If $f(x)\equiv 5$, what is the gradient of the graph of $f(x)$?

SUMMARY OF RESULTS.

Draw the graph of any function $y=f(x)$; see Figure 27.

Suppose $ON=x$ and $OM=x+h$, so that $PR=NM=h$.

Then $PN=f(x)$ and $QM=f(x+h)$.

$$\therefore \; QR=f(x+h)-f(x).$$

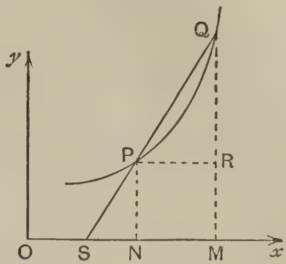

FIG. 27.

The average gradient of PQ is $\dfrac{f(x+h)-f(x)}{h}$.

And the gradient at P is $\underset{h\to 0}{\mathrm{Lt}}\dfrac{f(x+h)-f(x)}{h}$.

The gradient at P is the slope of the tangent at P to the curve.

CHAPTER III.

DIFFERENTIATION.

NOTATION.

FIGURE 28 represents the graph of any function $y=f(x)$.

If $ON=x$, then $PN=y=f(x)$.

Q is any point close to P on the curve and QM its ordinate.

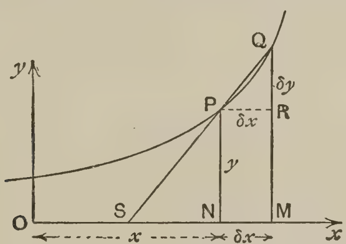

FIG. 28.

The length of NM is represented by the symbol δx, which means " a small increment of the variable x," or colloquially " a little bit of x."

And δy represents the *consequent* change in y; so that $\delta y = QM - PN = QR$.

Thus
$$ON=x; \quad OM=x+\delta x;$$
$$PN=y=f(x); \quad QM=y+\delta y=f(x+\delta x);$$
$$\therefore \delta y = f(x+\delta x)-f(x),$$
and
$$\frac{QR}{PR}=\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}.$$

Now $\dfrac{QR}{PR}$ = the gradient of the chord PQ.

$\therefore$ the gradient of the tangent at P

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$$

= rate of change of $f(x)$ with respect to x.

The expression $\underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x}$ is written $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}y$, and is called the *differential coefficient* of y with respect to x.

The expression $\underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x+\delta x) - f(x)}{\delta x}$ is written $\dfrac{df(x)}{dx}$ or $\dfrac{d}{dx}f(x)$, and is called the *differential coefficient* of $f(x)$ with respect to x.

The process of finding this limit is called "*differentiating with respect to x*," and the result is sometimes called the "*derived function*" of $f(x)$ and written $f'(x)$.

Summary.

(i) $\dfrac{d}{dx}f(x) = \underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x+\delta x) - f(x)}{\delta x}$.

(ii) If $y = f(x)$, $\dfrac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \dfrac{f(x+\delta x) - f(x)}{\delta x}$.

Example I. Differentiate $3x^2 + 7$ with respect to x.

$$\frac{d}{dx}(3x^2+7) = \underset{\delta x \to 0}{\text{Lt}} \frac{[3(x+\delta x)^2 + 7] - [3x^2 + 7]}{\delta x}$$

$$= \text{Lt} \frac{[3\{x^2 + 2x\,\delta x + (\delta x)^2\} + 7] - 3x^2 - 7}{\delta x}$$

$$= \text{Lt} \frac{6x\,\delta x + 3(\delta x)^2}{\delta x}$$

$$= \text{Lt}\,[6x + 3\,\delta x], \text{ provided } \delta x \neq 0,$$

$$= 6x.$$

Example II. Find $\frac{d}{dx}\left(\frac{1}{x}\right)$.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x} \right)$$

$$= \text{Lt}\, \frac{x - (x+\delta x)}{x(x+\delta x)\,.\,\delta x}$$

$$= \text{Lt} - \frac{\delta x}{(x^2 + x\,.\,\delta x)\,.\,\delta x}$$

$$= \text{Lt} - \frac{1}{x^2 + x\,.\,\delta x}, \text{ provided } \delta x \neq 0,$$

$$= -\frac{1}{x^2}.$$

EXERCISE III. a.

1. (i) Simplify $5(x+\delta x) - 5x$. (ii) Find $\frac{d}{dx}(5x)$.

2. (i) Simplify $(x+\delta x)^2 - x^2$. (ii) Find $\frac{d}{dx}(x^2)$.

3. (i) Simplify $[5(x+\delta x)^2 - 3(x+\delta x) + 7] - [5x^2 - 3x + 7]$.

(ii) Find $\frac{d}{dx}[5x^2 - 3x + 7]$.

4. (i) If $f(x) \equiv (1+x)^2$, what is $f(x+\delta x)$?

(ii) Find $\frac{d}{dx}(1+x)^2$.

5. Differentiate $x(1+x)$.

6. Differentiate (i) x^2; (ii) $3x^2$; (iii) $7x^2$, and write down the value of $\frac{d}{dx}(100x^2)$.

7. Differentiate (i) x^3; (ii) $4x^3$; (iii) $5x^3$; and write down the value of $\frac{d}{dx}(29x^3)$.

8. Given that $\underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^4 - x^4}{\delta x} = 4x^3$, write down the values of

(i) $\frac{d}{dx}(3x^4)$; (ii) $\frac{d}{dx}(10x^4)$.

9. Given that $\frac{d}{dx}x^5 = 5x^4$, write down the values of

(i) $\frac{d}{dx}(7x^5)$; (ii) $\frac{d}{dx}(x^5 + 7)$.

10. Given that $\frac{d}{dx}x^6 = 6x^5$, write down the values of

(i) $\frac{d}{dx}(\frac{3}{4}x^6)$; (ii) $\frac{d}{dx}(5 - x^6)$; (iii) $\frac{d}{dx}(x - 2x^6)$.

11. Express in the limit form the fact that $\frac{d}{dx}x^n = nx^{n-1}$.

12. Use the facts that

$$\underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^2 - x^2}{\delta x} = 2x \quad \text{and} \quad \underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^3 - x^3}{\delta x} = 3x^2$$

to write down the values of

(i) $\underset{\delta x \to 0}{\text{Lt}} \frac{[(x+\delta x)^3 + 2(x+\delta x)^2] - [x^3 + 2x^2]}{\delta x}$;

(ii) $\frac{d}{dx}(3x^2 + x^3)$; (iii) $\frac{d}{dx}(\frac{2}{3}x^3 + \frac{1}{2}x^2 + 5)$.

13. Express as limits

(i) $\frac{d}{dx}f(x)$; (ii) $\frac{d}{dx}\phi(x)$; (iii) $\frac{d}{dx}[3f(x) - 5\phi(x)]$.

If $\frac{d}{dx}f(x) = u$ and $\frac{d}{dx}\phi(x) = v$, what is $\frac{d}{dx}[3f(x) + 5\phi(x)]$?

14. Find

(i) $\frac{d}{dx}(3x)$; (ii) $\frac{d}{dx}(5x)$; (iii) $\frac{d}{dx}(3x \times 5x)$; (iv) $\frac{d}{dx}(3x + 5x)$.

Is $\frac{d}{dx}(3x \times 5x)$ equal to $\frac{d}{dx}(3x) \times \frac{d}{dx}(5x)$?

Is $\frac{d}{dx}(3x + 5x)$ equal to $\frac{d}{dx}(3x) + \frac{d}{dx}(5x)$?

15. (i) Express as limits $\frac{d}{dx}(3x^2)$; $\frac{d}{dx}(5x^3)$; $\frac{d}{dx}(3x^2 + 5x^3)$; $\frac{d}{dx}(3x^2 - 5x^3)$; $\frac{d}{dx}(3x^2 \times 5x^3)$; $\frac{d}{dx}(5x^3 \div 3x^2)$.

(ii) Is $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2)$?

(iii) Is $\frac{d}{dx}(3x^2 \pm 5x^3) = 3\frac{d}{dx}(x^2) \pm 5\frac{d}{dx}(x^3)$?

(iv) Is $\frac{d}{dx}(3x^2 \times 5x^3) = 3\frac{d}{dx}(x^2) \times 5\frac{d}{dx}(x^3)$?

(v) Is $\frac{d}{dx}(5x^3 \div 3x^2) = 5\frac{d}{dx}(x^3) \div 3\frac{d}{dx}(x^2)$?

16. What general formula covers the following facts ?

$$\frac{d}{dx}x^2=2x; \quad \frac{d}{dx}x^3=3x^2; \quad \frac{d}{dx}x^4=4x^3; \quad \frac{d}{dx}x^5=5x^4.$$

17. What general formula covers the following facts ?

$$\frac{d}{dx}(7x^2)=7\frac{d}{dx}(x^2); \quad \frac{d}{dx}(11x^4)=11\frac{d}{dx}(x^4); \quad \frac{d}{dx}\left(\frac{93}{x}\right)=93\frac{d}{dx}\left(\frac{1}{x}\right).$$

18. What general formula covers the following facts ?

$$\frac{d}{dx}(x^2+x^3)=\frac{d}{dx}(x^2)+\frac{d}{dx}(x^3); \quad \frac{d}{dx}(7x^2-11x^5)=\frac{d}{dx}(7x^2)-\frac{d}{dx}(11x^5).$$

19. Take simple functions for $f(x)$ and $\phi(x)$ to show that $\frac{d}{dx}[f(x)\times\phi(x)]$ is not equal to $\frac{d}{dx}f(x)\times\frac{d}{dx}\phi(x)$.

20. Write down special cases of the general formula

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

when n equals (i) 8 ; (ii) 50 ; (iii) -1 ; (iv) -3 ; (v) 1 ; (vi) 0 ; (vii) $\frac{3}{2}$; (viii) $\frac{1}{2}$; (ix) $-\frac{1}{2}$; (x) $-m$.

21. If $y=x^2$, find δy when $x=2$, $\delta x=0{\cdot}1$.

22. If $y=\frac{1}{x}$, find δy when $x=3$, $\delta x=\frac{1}{3}$.

23. If $y=x^3$, find δy when (i) $x=1$, $\delta x=0{\cdot}1$; (ii) $x=1$, $\delta x=0{\cdot}01$.

24. If $y=3x^2$, find δx when $x=2$, $\delta y=0{\cdot}1$.

25. If $y=x^2+x$, find δy when $x=10$, $\delta x=-1$.

Summary of Results.

(i) $\frac{d}{dx}(x^n)=nx^{n-1}$, where n is integral or fractional, positive or negative.

(ii) $\frac{d}{dx}(C)=0$, if C is any constant, *i.e.* a number independent of x.

(iii) $\frac{d}{dx}(Cx^n)=C\frac{d}{dx}(x^n)$
$=Cnx^{n-1}$, where C is any constant.

(iv) $\frac{d}{dx}[f(x)\pm\phi(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[\phi(x)]$.

Example III. Find $\dfrac{d}{dx}(x)$; $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$; $\dfrac{d}{dx}(\sqrt{x})$.

Since $$\frac{d}{dx}(x^n)=nx^{n-1}$$

(i) Put $n=1$, $\therefore \dfrac{d}{dx}x=1x^{1-1}=x^0=1$,

or, more simply, $\dfrac{d}{dx}(x)=\underset{\delta x\to 0}{\mathrm{Lt}}\dfrac{(x+\delta x)-x}{\delta x}=\mathrm{Lt}\dfrac{\delta x}{\delta x}=1.$

(ii) Put $n=-2$, $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)=\dfrac{d}{dx}(x^{-2})=-2x^{-2-1}=-2x^{-3}=-2\times x^{-3}$

$$=-2\times\frac{1}{x^3}=-\frac{2}{x^3}.$$

iii) Put $n=\frac{1}{2}$, $\dfrac{d}{dx}(\sqrt{x})=\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}$

$$=\frac{1}{2}\times\frac{1}{x^{\frac{1}{2}}}=\frac{1}{2}\times\frac{1}{\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Example IV. Find $\dfrac{d}{dx}\left(3x^4-7x+5-\dfrac{2}{x}\right)$.

$$\begin{aligned}\text{The expression}&=\frac{d}{dx}(3x^4)-\frac{d}{dx}(7x)+\frac{d}{dx}(5)-\frac{d}{dx}\left(\frac{2}{x}\right)\\&=3\frac{d}{dx}(x^4)-7\frac{d}{dx}(x)+0-2\frac{d}{dx}(x^{-1})\\&=3(4x^3)-7(1)+0-2(-1)(x^{-1-1})\\&=12x^3-7+2x^{-2}\\&=12x^3-7+\frac{2}{x^2}.\end{aligned}$$

After a little practice, most of the intermediate steps in the working can be omitted.

SUCCESSIVE DIFFERENTIATION.

If $y=x^4$, $$\frac{dy}{dx}=4x^3;$$

$$\therefore\ \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(4x^3)=4\frac{d}{dx}(x^3)=4\times 3x^2=12x^2.$$

For the sake of brevity, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is written $\dfrac{d^2y}{dx^2}$: this symbol is called the second differential coefficient of y with respect to x.

Similarly $\dfrac{d^2s}{dt^2}$ is short for $\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right)$.

Example V. Find $\dfrac{d^2}{dx^2}(3x^2+4x-7)$.

$$\frac{d}{dx}(3x^2+4x-7)=\frac{d}{dx}(3x^2)+\frac{d}{dx}(4x)-\frac{d}{dx}(7)=6x+4;$$

$$\therefore \frac{d^2}{dx^2}(3x^2+4x-7)=\frac{d}{dx}(6x+4)=6.$$

EXERCISE III. b.

Differentiate with respect to x the expressions in Examples 1-30.

1. x^7. 2. $10x^3$. 3. $x+\dfrac{1}{x}$. 4. $\dfrac{1}{x^5}$.

5. $\dfrac{3}{x}$. 6. $\dfrac{2}{x^2}$. 7. $3x^2-2x$. 8. $\frac{1}{4}x^4-2$.

9. $x^2\times x^3$. 10. $(5x)^3$. 11. $(1-x)^2$. 12. $\dfrac{1}{x^n}$.

13. ax. 14. x^a. 15. $\dfrac{c}{x^3}$. 16. $b\sqrt{x}$.

17. $\sqrt{x^3}$. 18. $\dfrac{1}{\sqrt{x}}$. 19. x^0. 20. $\sqrt[3]{x}$.

21. $\dfrac{2}{3}x^3-\dfrac{x}{5}$. 22. $\dfrac{x}{10}+\dfrac{10}{x}$. 23. $7x^3-2x-5$. 24. $\dfrac{4}{3}\pi x^3$.

25. $3x^4-\dfrac{1}{2}x+7-\dfrac{6}{x}$. 26. $5x^4+3x^2+\dfrac{1}{x}-\dfrac{1}{x^2}$.

27. $(x^2+1)(x+2)$. 28. $\left(x+\dfrac{1}{x}\right)^2$. 29. x^{2n}. 30. x^{-k}.

31. If $y=3x^2+5$, find $\dfrac{dy}{dx}$ when $x=1$.

32. If $y=1+2x-x^2$, find $\dfrac{dy}{dx}$ when $x=-2$.

33. If $y=2x^3-9x^2+12x$, find the values of x for which $\dfrac{dy}{dx}=0$.

34. If $y=7x^3-9x$, find $\dfrac{d^2y}{dx^2}$. 35. If $y^2=x$, find $\dfrac{d^2y}{dx^2}$.

36. If $y=6x^2$, prove that $x\dfrac{dy}{dx}=2y$.

37. If $y=x(1-x)$, prove that $1+\dfrac{dy}{dx}=\dfrac{2y}{x}$.

38. If $y=x^3$, prove that $\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=18y$.

39. If $s=100t-16t^2$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ when $t=1$ and $t=0$.

40. If $y=7x^4$, express (i) $2y\cdot\frac{dy}{dx}$; (ii) $\frac{d}{dx}(y^2)$ in terms of x.

41. If $y=2z^2$, $z=3+5x$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

42. If $y=z+z^3$, $z=1+x^2$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

43. The cubical elasticity of a fluid is equal to $-v\cdot\frac{dp}{dv}$ where the volume v and the pressure p are connected by the equation $pv=c$ (a constant). Simplify this expression.

44. The radius of the circle which approximates most closely to the shape of the curve $y=\frac{2x^2}{3}$ at any point is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2};$$

find its value when $x=1$.

TURNING POINTS.

A function $f(x)$ is called an "*increasing function*" of x for any range of values of x in which, as x increases, the value of $f(x)$ also increases.

It is called a "*decreasing function*" of x if, as x increases, the value of $f(x)$ decreases.

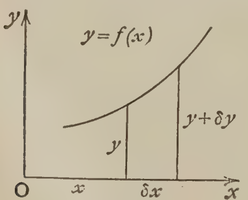

FIG. 29.

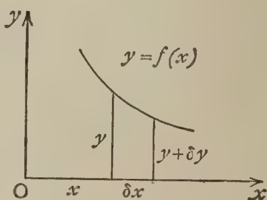

FIG. 30.

Figure 29 represents an increasing function.
Figure 30 represents a decreasing function.

If δx is positive, then δy is positive for an increasing function (Figure 29), and δy is negative for a decreasing function (Figure 30).

Since $\frac{dy}{dx} = \underset{x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x}$, we see that $\frac{dy}{dx}$ is positive for an increasing function, and is negative for a decreasing function.

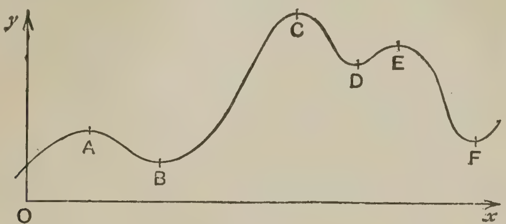

FIG. 31.

Figure 31 represents a function which is an increasing function for some ranges of values of x, and a decreasing function for others.

Thus $f(x)$ is a decreasing function from A to B, and an increasing function from B to C.

The separating point B is called a *turning point*, and the value of $f(x)$ corresponding to the point B is called a *turning value* of the function.

There are two kinds of turning points.

A, C, E, ... correspond to values of x for which the function is greater than at any other point near it: at these points the function is said to be a *maximum*.

B, D, F, ... correspond to values of x for which the function is less than at any other point near it: at these points the function is said to be a *minimum*.

At a turning point, $\frac{dy}{dx} = 0$, for the function is neither increasing $\left(i.e.\ \frac{dy}{dx} \text{ positive}\right)$ nor decreasing $\left(i.e.\ \frac{dy}{dx} \text{ negative}\right)$.

EXERCISE III. c.

1. Fig. 32 is the graph of $y=x^2$. Is y an increasing or decreasing function (i) from A to O, (ii) from O to B? Is $\frac{dy}{dx}$ positive or negative (α) from A to O, (β) from O to B? Has y a maximum or minimum value anywhere?

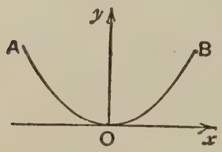

FIG. 32.

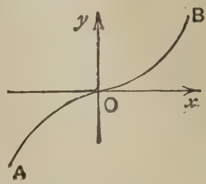

FIG. 33.

2. Fig. 33 is the graph of $y=x^3$. Answer the same questions as in Ex. 1.

3. Fig. 34 is the graph of $y=4+3x-x^2$. Is y an increasing or decreasing function from (i) A to B, (ii) B to C, (iii) C to D, (iv) D to E, (v) E to F? What are the signs of $\frac{dy}{dx}$ for these five portions? Has y a maximum or minimum value anywhere?

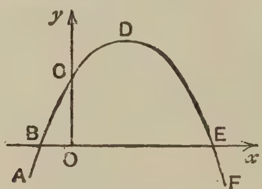

FIG. 34.

4. Draw freehand the graph of a function $y=f(x)$ which starts from the origin O, and is such that

(i) from O to A, y is a decreasing function;

(ii) from A to B, $\frac{dy}{dx}$ is positive and y is negative;

(iii) from B to C, y is a positive increasing function;

(iv) from C to D, $\frac{dy}{dx}$ is negative and y is positive;

(v) from D to E, $\frac{dy}{dx}$ is negative and y is negative;

(vi) from E to F, $\frac{dy}{dx}$ is positive.

Has the function any maximum or minimum values?

5. Draw freehand the graph of a function $y=f(x)$ for which

(i) x is negative, y is negative, $\dfrac{dy}{dx}$ is positive;

(ii) x is negative, y is negative, $\dfrac{dy}{dx}$ is negative;

(iii) x is negative, y is positive, $\dfrac{dy}{dx}$ is positive;

(iv) x is negative, y is positive, $\dfrac{dy}{dx}$ is negative.

6. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\dfrac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $1, \frac{1}{2}, 0, -\frac{1}{2}, -1$.

(ii) What kind of a point is C?

(iii) Is $\dfrac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ or $\dfrac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

7. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\dfrac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

(ii) What kind of a point is C?

(iii) Is $\dfrac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\dfrac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

8. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\dfrac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively $1, \frac{1}{2}, 0, \frac{1}{2}, 1$.

(ii) Is C a turning point?

(iii) Is $\dfrac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\dfrac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

9. Answer the various questions in Ex. 8, taking the values of $\frac{dy}{dx}$ at A, B, C, D, E to be respectively $-1, -\frac{1}{2}, 0, -\frac{1}{2}, -1$.

10. AB, CD, EF, GH are portions of the graph of $y=f(x)$.

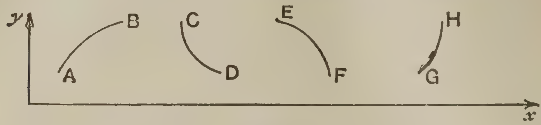

FIG. 35.

What can you say about the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the arcs

(i) AB; (ii) CD; (iii) EF; (iv) GH?

11. Can one of the minimum values of a function be greater than one of its maximum values? Illustrate by a figure.

12. Can a function have (i) exactly one minimum and two maximum values, (ii) exactly one minimum and three maximum values? Illustrate by a figure.

13. Part of the graph of the function $y=f(x)$ is represented by the curve $ACBDPQ$ in Fig. 26, p. 30; copy this freehand, and underneath it draw roughly the corresponding portions of the graphs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

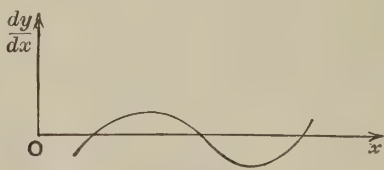

FIG. 36.

14. Fig. 36 represents part of the graph of $\frac{dy}{dx}$; draw roughly the corresponding portions of the graph of y and of $\frac{d^2y}{dx^2}$.

15. Part of the graph of the function $y=f(x)$ is represented by the curve $ABCDEF$ in Fig. 31, p. 43; show in tabular form the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the various portions of the graph, and state at which points either $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ is zero.

16. Fig. 37 shows the graph of $y=x^2$; the tangent at P meets Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=y$; prove that (i) $\dfrac{PN}{NT}=$ the slope at $P=2x$; (ii) $OT=TN$.

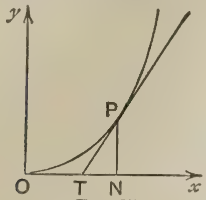

FIG. 37.

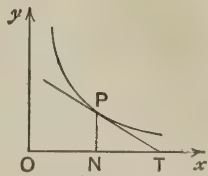

FIG. 38.

17. Fig. 38 shows the graph of $y=\dfrac{1}{x}$; the tangent at P meets Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=y$; prove that (i) $\dfrac{PN}{NT}=-$ the slope at $P=\dfrac{1}{x^2}$; (ii) $ON=NT$.

18. If Fig. 37 represents the graph of $y=3x^2$ and if P is the point (2, 12), find the length of OT.

19. If Fig. 38 represents the graph of $y=\dfrac{8}{x^2}$ and if P is the point (2, 2), find the length of OT.

20. If Fig. 37 represents the graph of $y=x^3$, prove that

$$OT=\tfrac{2}{3}ON.$$

21. If Fig. 38 represents the graph of $y^2=\dfrac{1}{x^3}$, prove that

$$ON=\tfrac{3}{5}\,.\,OT.$$

Summary of Results.

For the function $y=f(x)$,

(i) $\dfrac{dy}{dx}=0$, both at a maximum and a minimum.

(ii) $\dfrac{d^2y}{dx^2}$ is negative at a maximum;

$\dfrac{d^2y}{dx^2}$ is positive at a minimum.

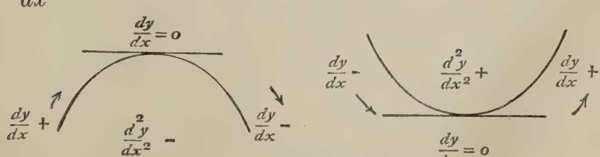

FIG. 39.

(iii) $\frac{dy}{dx}$ changes from $+$ to $-$ in passing through a maximum;

$\frac{dy}{dx}$ changes from $-$ to $+$ in passing through a minimum;

or $\frac{dy}{dx}$ is a decreasing function in passing through a maximum;

$\frac{dy}{dx}$ is an increasing function in passing through a minimum;

(iv) If $\frac{dy}{dx}=0$ but does not change sign, there is neither a maximum nor a minimum.

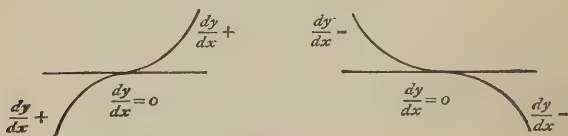

FIG. 40.

MAXIMA AND MINIMA PROBLEMS.

Example VI. The strength of a beam of uniform rectangular section varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam that can be cut from a cylindrical tree-trunk of diameter 20 inches.

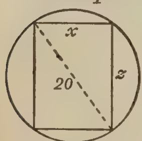

FIG. 41.

Let the breadth be x inches and the depth z inches;

$$\therefore\ x^2+z^2=20^2 \quad \text{(Pythagoras)}$$
$$=400.$$

Now the strength varies as xz^2

$$=kxz^2=kx(400-x^2), \text{ where } k \text{ is a constant,}$$
$$=k(400x-x^3);$$

$\therefore$ the function $y=400x-x^3$ is to be a maximum.

$$\frac{dy}{dx}=400-3x^2;$$

$$\therefore\ \frac{dy}{dx}=0 \text{ if } 400-3x^2=0 \quad \text{or} \quad x^2=\frac{400}{3}=133{\cdot}3;$$

$$\therefore\ x=\pm 11{\cdot}54.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}(400-3x^2)=-6x;$$

$$\therefore\ \frac{d^2y}{dx^2} \text{ is negative if } x \text{ is positive;}$$

$$\therefore\ x=11{\cdot}54 \text{ gives a } \textit{maximum} \text{ value for } y;$$

$\therefore$ the required breadth is 11·5 inches.

Note.—Sometimes it is necessary to find the maximum or minimum values of expressions which are not in a form that can be differentiated by the rules already considered. It may be possible however, either to change the form by substituting or to avoid the difficulty in some other way.

Example VII. Find (i) the value of x for which $\sqrt{x^2-8x+21}$ is a minimum, (ii) the value of x for which $\dfrac{x-1}{(x+1)^2}$ is a maximum.

(i) $\sqrt{x^2-8x+21}$ has its least value if $x^2-8x+21$ is a minimum;

$$\therefore\ \frac{d}{dx}(x^2-8x+21)=0;$$

$$\therefore\ 2x-8=0 \quad\text{or}\quad x=4.$$

It is a minimum because $\dfrac{d^2}{dx^2}(x^2-8x+21)=2$ and is therefore positive.

(ii) Let $y=\dfrac{x-1}{(x+1)^2}$ and put $x+1=z$;

$$\therefore\ y=\frac{z-2}{z^2}=\frac{1}{z}-\frac{2}{z^2};$$

$$\therefore\ \frac{dy}{dz}=-\frac{1}{z^2}+\frac{4}{z^3}=\frac{(-z+4)}{z^3};$$

$$\therefore\ \frac{dy}{dz}=0 \text{ if } z=4 \quad\text{or}\quad x+1=4 \quad\text{or}\quad x=3.$$

Also $$\frac{d^2y}{dz^2}=\frac{2}{z^3}-\frac{12}{z^4}=\frac{2z-12}{z^4}=\frac{-4}{4^4} \text{ if } z=4;$$

$$\therefore\ \text{for } z=4,\ y \text{ is a maximum, since } \frac{d^2y}{dz^2} \text{ is negative;}$$

$$\therefore\ \text{for } x=3,\ \frac{x-1}{(x+1)^2} \text{ is a maximum.}$$

EXERCISE III. d.

1. Find the values of x which correspond to turning values of the following functions; determine whether they are maxima or minima; and sketch roughly the graphs of the functions:

(i) x^2-2x; (ii) x^2+4x+3; (iii) $3+8x-10x^2$;
(iv) $x+\frac{1}{x}$; (v) $x-\frac{1}{x}$; (vi) x^4-4x;
(vii) x^4-x^3; (viii) x^3-x^2-x+1; (ix) x^3-3x^2-9x+7;
(x) x^3-3x^2+3x-1.

2. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long.

3. The parcel post regulations require that the sum of the length and girth of a parcel shall not exceed 6 feet. Find the volume of the largest box with a square base that can be sent by post.

4. A closed rectangular cistern is to be constructed to contain 80 cu. feet. It is to be 5 feet long. Find the breadth when the *total* area of its surface is a minimum.

5. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam which has a perimeter of 4 feet.

6. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long, if an existing fence is utilised to form one side.

7. A box without a lid is to be made from a sheet of metal of negligible thickness and is to have square ends; it is to hold $4\frac{1}{2}$ cu. feet. What is the least area of metal required?

8. For a steamer travelling v knots, the cost of the coal is £$\frac{v^2}{18}$ per hour, and other expenses amount to £8 per hour. What is the most economical speed for a journey of 200 nautical miles?

9. An open gutter of rectangular section is formed out of a long rectangular strip of sheet iron 9 feet wide. Find the maximum area of the cross-section.

10. A is 8 miles north and B is 6 miles east of a point O. Two men, starting at the same time from A and B, walk towards O at 4 miles an hour. If their distance apart is x miles after t hours, prove that $x^2=32t^2-112t+100$, and find when they are nearest together.

11. A rectangular sheet of cardboard is 8″ long and 5″ wide. Equal squares are cut out at each of the corners and the remainder

is folded so as to form an open box. Find the maximum volume of the box.

12. Given $\pi r^2 h = 5$, find the value of r for which $2\pi r^2 + 2\pi rh$ is a minimum. What geometrical problem corresponds to this question?

13. A particle projected in a resisting medium is finally brought to rest: it travels s feet in t seconds, where $s = 6t - \frac{1}{2}t^3$. How far does it go?

14. Using the data of Ex. 3, find the volume of the largest circular cylinder that can be sent by parcel post.

15. If a very thin rod one foot long swings like a pendulum, the expression $x(1-x)^2$ measures the tendency to break at a place x feet from the point of suspension. Find where the rod is most likely to break.

16. The perimeter of a sheet of metal in the form of a circular sector is 1 foot. For what radius is the area a maximum?

17. From a circular sheet of paper a sector is removed and the remainder is folded to form a circular cone. What fraction must be removed in order to give the cone of maximum volume?

18. A rod one foot long is cut into two pieces to form the hypotenuse and one side of a right-angled triangle. How must it be cut to give the triangle of greatest area?

19. The efficiency of a screw of certain material is $\dfrac{4h - 3h^2}{3 + 4h}$, where h is the tangent of its pitch. What is the greatest efficiency? [Put $3 + 4h = x$.]

20. Assuming that the rigidity of a rectangular beam varies as its breadth and the cube of its depth, find the breadth of the most rigid beam that can be cut from a cylindrical trunk of diameter 3 feet.

21. A cylindrical vessel is open at one end and closed at the other; for a given surface, prove that the volume is greatest if its height equals the radius of the base.

22. A skeleton box with two square ends is formed with 12 pieces of wire, and four other pieces of wire form an equal square round the middle of it. The total amount of wire available is one yard. What is the maximum volume of the box?

23. A piece of wire 2 feet long is cut into two parts, one of which is bent to form a square and the other a circle. If the sum of the areas is a minimum, find the radius of the circle.

24. If $2x + y = 5$, what is the greatest value of $x^2 + 3xy + y^2$?

25. What is the greatest value of $\dfrac{x}{x^2 + 4}$?

26. What is the greatest value of $\dfrac{x}{(x^2 + 1)^2}$?

27. Find the minimum value of $\sqrt{\frac{x}{a}+\frac{a}{x}}$.

28. Find the radius of a circular cylinder which is cut from a sphere of radius 10 inches so that

(i) the volume of the cylinder is a maximum ;
(ii) the curved surface of the cylinder is a maximum.

29. P is a variable point on the line ABC ; $AB=2''$, $BC=3''$. Find the position of P for which $PA^2+PB^2-PC^2$ is a minimum.

30. ABC is a triangular field right-angled at A ; P, Q are points on AB, AC such that a fence from P to Q bisects the field. If $AB=a$, $AC=b$, $AP=x$, $PQ=y$ yards, express y as a function of x. If $a=625$, $b=800$, find the length of AP in order that the fence may be as short as possible; find also the length of the fence. (C.S.C.)

31. $ABCD$ is a rectangular sheet of cardboard (see Fig. 42) from which the shaded portions are removed. The remainder is used to make a closed box, as shown in the figure. If $AB=a$, $BC=b$, $AP=x$ inches, find the volume of the box in terms of a, b, x ; if $a=6$, $b=12$, find the value of x for which the volume is a maximum. (C.S.C.)

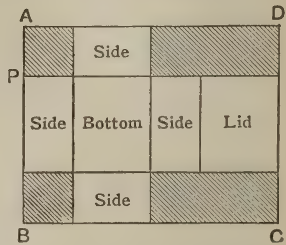

FIG. 42.

APPROXIMATION.

Figure 43 represents the graph of $y=f(x)$.

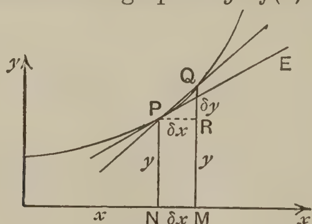

FIG. 43.

With the usual notation,

the slope of the chord PQ is $\frac{\delta y}{\delta x}$,

the slope of the tangent PE is $\frac{dy}{dx}$.

These are obviously not equal to each other.

But $\frac{dy}{dx}$ does equal the limit of $\frac{\delta y}{\delta x}$, when $\delta x \to 0$.

And we may say that

$$\frac{dy}{dx} \simeq \frac{\delta y}{\delta x}, \text{ when } \delta x \text{ is small,}$$

or

$$\delta y \simeq \frac{dy}{dx} \times \delta x, \text{ when } \delta x \text{ is small.}$$

The smaller δx is, the nearer the slope of the chord PQ is to the slope of the tangent PE.

Example VIII. If $y = x^2$, what is the error in the approximation $\delta y \simeq \frac{dy}{dx} \times \delta x$?

Now $$y + \delta y = (x + \delta x)^2;$$

$$\therefore\ y + \delta y = x^2 + 2x\,.\,\delta x + (\delta x)^2,$$

but $$y = x^2;$$

$$\therefore\ \delta y = 2x\,.\,\delta x + (\delta x)^2.$$

But $$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x;$$

$$\therefore\ \delta y = \frac{dy}{dx} \times \delta x + (\delta x)^2;$$

$$\therefore\ \delta y \simeq \frac{dy}{dx} \times \delta x \text{ with error } (\delta x)^2.$$

The relative size of the error may be seen in a figure.

If $y = x^2$, y sq. inches is the area of a square of side x inches, and $(y + \delta y)$ sq. inches is the area of a square of side $(x + \delta x)$ inches.

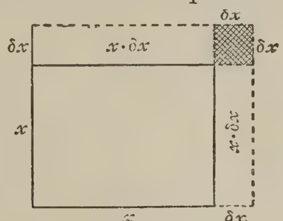

FIG. 44.

In taking δy equal to $\frac{dy}{dx} \times \delta x$ or $2x\,\delta x$, we are supposing its value is the sum of the two rectangles and neglecting the small shaded square in Figure 44.

Suppose x is 1″ and δx is 0·1″, the error is 0·01 sq. in.

EXERCISE III. e.

1. Find the error in the relation $\delta y \simeq \frac{dy}{dx} \times \delta x$ if

(i) $y = 5x^2$; (ii) $y = x^3$.

2. The area of an isosceles right-angled triangle of side x inches is A sq. ins.

(i) Find geometrically the value of δA in terms of δx.

(ii) Express A in terms of x, and evaluate $\delta A - \frac{dA}{dx} \times \delta x$.

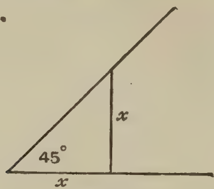

FIG. 45.

3. (i) Find an approximate expression for the increase in area of a circle when the radius increases from r to $r + \delta r$.

(ii) Illustrate the result geometrically.

(iii) Find approximately the difference in area of two circles of radii 10″ and 10·01″.

4. The radius of a spherical soap bubble increases from 1″ to 1·01″; find approximately its change of volume.

5. A body travels s feet in t sec., where $s = 10t - \frac{1}{5}t^3$; find an approximate expression (i) for δs in terms of δt, (ii) for the distance it moves in the interval of time $t = 2$ to $t = 2{\cdot}1$.

6. A gas at constant temperature under a pressure of p lb. per sq. inch occupies v cu. inches, where $pv = c$ (a constant); if the pressure is increased from p to $p + \delta p$, find an approximate expression for the change of volume.

7. Two telegraph posts of equal height are at a distance

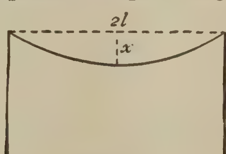

FIG. 46.

$2l$ feet apart. When the sag at the mid-point is x feet, the length of wire exceeds the distance between the poles by y feet, where

$y=\dfrac{4x^2}{3l}$. Find an approximate value for δy in terms of δx. What does this mean? If the distance between the poles is 30 yards, what is the effect of increasing the sag from 10 inches to 11 inches?

8. At sea-level, water boils at 212° F. At a height h feet above sea-level, the boiling point is lowered t degrees, where $h=520t+t^2$; find approximately the difference of heights of two places where the boiling points are 200° and 201° F. respectively.

9. The perimeter of a circular pond is measured as 1321 yards; assuming this is correct to the nearest yard, what error may be expected in the area of the pond when calculated from this result?

10. A beam AB 30 feet long supported at each end is just strong enough to carry a load of M tons placed at a point P on the beam such that $M=\dfrac{1}{10}\left(k+\dfrac{1}{k}+\dfrac{1}{2}\right)$, where $\dfrac{AP}{PB}=k$. Express δM in terms of k and δk. What does this mean? Find approximately what change in the load may be necessary when it is shifted from a point 5 feet from A to a point 6 feet from A?

11. Fig. 47 represents the graph of $y=x^2$, unit 1″ on each axis. If $ON=x$, it can be proved that the area bounded by ON, NP and the arc OP is $A=\frac{1}{3}x^3$ sq. in. Find approximately δA in terms of δx, and interpret the result geometrically.

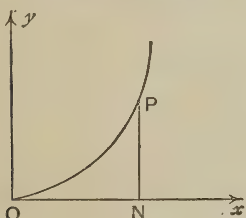

FIG. 47.

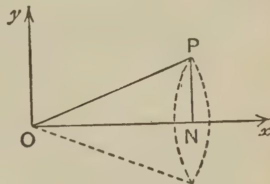

FIG. 48.

12. In Fig. 48 OP is the graph of $y=\frac{1}{2}x$; a circular cone is obtained by revolving OP about Ox. If $ON=x$ and the volume of the cone is V, prove that $V=\frac{1}{12}\pi x^3$. Find approximately δV in terms of δx, and interpret the result geometrically.

13. The bowl of a wine-glass is formed by revolving the graph of $y=x^2$ (see Fig. 47) about Oy. It can be proved that when the depth of wine in the glass is y inches, the volume is $V=\frac{1}{2}\pi y^2$ cu. in. Find approximately δV in terms of δy, and interpret the result geometrically.

14. If the depth of water in a hemispherical bowl of radius a in. is x in., the volume of the water is $V=\pi x^2\left(a-\frac{x}{3}\right)$ cu. in. Find approximately δV in terms of δx, and interpret the result geometrically.

15. If $y=z^5$ and $z=1+3x$, (i) prove that $\delta y \simeq 5z^4 \,.\, \delta z$; (ii) express δz in terms of δx; (iii) express δy in terms of x and δx; (iv) hence find $\frac{dy}{dx}$ and $\frac{d}{dx}(1+3x)^5$.

16. If $y=\sqrt{z}$ and $z=1+x^2$, (i) express δy in terms of δz and δz in terms of δx, and so find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}(\sqrt{1+x^2})$.

17. If $y=z^3$ and $z=x^3-x+7$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}[(x^3-x+7)^3]$.

18. If $y=\frac{1}{z^2}$ and $z=3x-5$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}\left[\frac{1}{(3x-5)^2}\right]$.

19. If $u=y^2$, and if y is a function of x, (i) find a relation between δu, δy, y; (ii) hence express $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$; (iii) prove that $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$.

20. If V cu. in. is the volume of a sphere of radius r in., then $V=\frac{4}{3}\pi r^3$. The radius of a sphere, which is expanding, is r in. after t sec.; (i) express δV in terms of r and δr; (ii) find a relation between $\frac{dV}{dt}$, $\frac{dr}{dt}$ and r; (iii) interpret this result.

RATE OF CHANGE.

Example IX. A vessel is in the shape of a circular cone of semi-vertical angle 45°. Water is poured into it at the rate

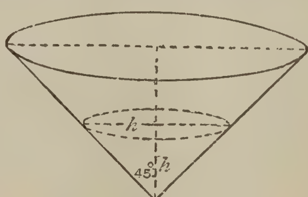

FIG. 49.

of 10 cu. in. per sec. At what rate is the level rising after 4 seconds?

When the depth is h in., the surface of the water is a circle of radius h in. ;

$\therefore$ the volume of water in the vessel is

$$V = \tfrac{1}{3}\pi h^2 \times h = \tfrac{1}{3}\pi h^3 \text{ cu. in.};$$

$$\therefore \frac{dV}{dh} = \tfrac{1}{3}\pi \times 3h^2 = \pi h^2;$$

$$\therefore \delta V \simeq \pi h^2 . \delta h.$$

Suppose the volume increases from V to $V + \delta V$ in δt secs.

Then $$\delta V = 10 . \delta t;$$

$$\therefore \pi h^2 . \delta h \simeq 10 . \delta t;$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi h^2}.$$

Now, when $t = 4$, $V = 40$;

$$\therefore \tfrac{1}{3}\pi h^3 = 40;$$

$$\therefore h^3 = \frac{120}{\pi};$$

$$\begin{array}{r} 2\cdot0792 \\ 0\cdot4971 \\ \hline 3 \,|\, 1\cdot5821 \\ \hline 0\cdot5274 \end{array}$$

$$\therefore h = \sqrt[3]{\frac{120}{\pi}} = 3\cdot368;$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi(3\cdot368)^2}$$

$$= 0\cdot2806;$$

$$\begin{array}{r} 1\cdot0000 \\ \bar{1}\cdot5519 \\ \hline \bar{1}\cdot4481 \end{array} \qquad \begin{array}{r} 0\cdot5274 \\ 2 \\ \hline 1\cdot0548 \\ 0\cdot4971 \\ \hline 1\cdot5519 \end{array}$$

$\therefore$ the level is rising at $0\cdot28''$ per sec. approximately.

EXERCISE III. f.

1. The temperature of a metal cube is being raised steadily so that each edge expands at the rate of $0\cdot01$ inch per hour. At what rate is the volume increasing when the edge is 2 inches ?

2. The cross-section of a trough is an isosceles right-angled triangle : the trough is 10 feet long. Water is poured into it at the rate of 5 cu. feet per sec. Find the rate at which the level is rising after 8 seconds.

3. The area of a circular ink-blot starts from zero and grows at the rate of 4 sq. inches per sec. ; at what rate is the radius increasing (i) when the radius is 1 in., (ii) after 3 seconds ?

4. A man 6 ft. high walks towards an electric arc street light at $3\frac{3}{4}$ miles an hour ; the light is 24 feet above the level of the street. Find the rate in feet per sec. at which his shadow diminishes in length.

5. A certain quantity of gas is contained in a spherical envelope of volume v cu. in.: the pressure is p lb. per sq. inch, where $pv=25$. The radius of the envelope increases at the rate of 2 in. per minute; find the rate at which the pressure is altering when the radius of the envelope is 5 inches.

6. A wine-glass is shaped so that when the depth of wine in it is y inches, its volume is $\frac{1}{5}y^4$ cu. in.; wine is poured in at the rate of 2 cu. in. per sec.; at what rate is the level rising when the depth is 2 in.?

7. The candle-power C of an incandescent lamp and its voltage V are connected by the equation $C=\frac{5V^6}{10^{11}}$. Find an expression for the rate of change of candle-power per unit increase of voltage, and evaluate it when $V=100$.

8. ABC is a triangle right-angled at C; P is a point on AB; $PNCM$ is a rectangle with its corners N, M on CA, CB; $CA=3$, $CB=4$, $AP=x$ inches; P moves from A to B at 2 in. per minute; at what rate is the area of the rectangle increasing after (i) 60 sec., (ii) 75 sec., (iii) 90 sec.?

9. A cube is expanding so that its volume after t minutes is $1000+0{\cdot}2t+0{\cdot}01t^2$ cu. inches; at what rate is its edge increasing in length after 10 minutes?

10. Sand is dropped on the ground at a steady rate of 10 cu. in. per sec. and forms a conical pile whose height remains equal to the radius of its base; at what rate is the height increasing after 5 seconds?

11. A bowl 5 inches deep is shaped so that when the depth of water in it is x inches, the amount of water is $8x+x^2$ cu. inches. Water is poured into it at the rate of 4 cu. in. per sec.; at what rate is the level rising when the depth is 3 inches?

12. The volume of a spherical envelope is increasing at the rate of 2 cu. cm. per sec.; at what rate is the area of the surface of the envelope stretching when the volume is 100 cu. cm.?

13. The volume of a circular cylinder is constant and equal to 150 cu. cm.; its height increases at the rate of 1 mm. a sec.; at what rate is its radius altering when its height is 3 cm.?

14. The heat required to raise the temperature of 1 gram of water from 0° C. to t° C. is Q units, where $Q=t+\frac{2}{10^5}t^2+\frac{3}{10^7}t^3$. The specific heat is the rate of increase of the quantity of heat per degree rise in temperature. Find the specific heat at 60° C.

15. Water runs out of a bath at a rate proportional to the amount in the bath at any moment. Originally the bath contains

30 cu. feet of water; if t sec. later Q cu. feet have run out, express the data of the question by an equation.

Summary of Results.

(i) An approximate value for

$$f(x+\delta x)-f(x) \quad \text{is} \quad \frac{d}{dx}f(x)\times\delta x.$$

(ii) If in calculating the value of $f(x)$ there is a small error δx in the value substituted for x, the error in the result is approximately $\frac{d}{dx}f(x)\times\delta x$.

(iii) If a distance x feet is described in t seconds, the speed is $\frac{dx}{dt}$ feet per second.

If y is a function of x and x is a function of t, the rate at which y increases is given by

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt},$$

for

$$\delta y \simeq \frac{dy}{dx}\times\delta x \quad \text{and} \quad \delta x \simeq \frac{dx}{dt}\times\delta t.$$

(iv) The formula $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ is used to differentiate complicated functions;

$$\textit{e.g.}\ \text{if } y=\sqrt{x} \quad \text{and} \quad x=3t^2-5,$$

then

$$y=\sqrt{3t^2-5},$$

and

$$\frac{d}{dt}(\sqrt{3t^2-5})=\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$=\frac{d}{dx}(\sqrt{x})\times\frac{d}{dt}(3t^2-5)$$

$$=\frac{1}{2\sqrt{x}}\times 6t$$

$$=\frac{3t}{\sqrt{3t^2-5}}.$$

REVISION PAPERS. K. 1–12.

K. 1.

1. Soundings are made on a sea-shore at low tide to discover the shape of the sea-bottom, and the following measurements are recorded:

Distance from low water mark in yards	50	100	150	200	250	300
Depth of water in fathoms	0·7	2	4	4·5	5	5·1

Draw a graph to illustrate these figures.

Estimate the average gradient of the sea-bottom

(i) For the first 300 yards out.

(ii) Between points 100 yards and 250 yards out.

(iii) Draw a tangent and find the gradient at a point 150 yards out. [1 fathom=6 feet.]

2. Find the value of (i) $\underset{h\to 0}{\mathrm{Lt}}\ \frac{1}{h}\{(x+h)^2+4(x+h)-x^2-4x\}$.

(ii) $\underset{h\to 0}{\mathrm{Lt}}\ \frac{1}{h}\left\{\frac{1}{x+h-1}-\frac{1}{x-1}\right\}$.

3. Find, by any method, the gradient of the graph of the function x^3-5x

(i) At the point where $x=1$.

(ii) At the point where $x=a$.

4. Sketch roughly a graph of the function $(x-2)^3(x-4)$.

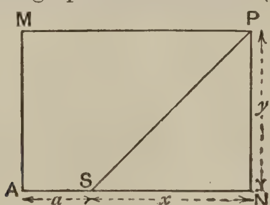

FIG. 50.

5. In Fig. 50 AS is a constant length a. P is a variable point, $PNAM$ is a rectangle such that $PS=PM$. If $PN=y$ and $NS=x$, find an equation connecting x and y, and express y as a function of x.

K. 2.

1. Draw roughly with the same axes, the graphs of

(i) $y=3-x$. (ii) $y=5-x$. (iii) $y=5$.

What are the gradients of these lines ?

2. Differentiate the following functions with respect to x:

(i) $4x-3x^2$. (ii) $3+4x-3x^2$.

(iii) $6+4x-3x^2$. (iv) $\frac{3}{2}x+\frac{2}{5}x^2$.

(v) $6x^{-1}$. (vi) $8\sqrt{x}$.

3. (i) Find the gradient of the graph $y=7+6x+3x^2$ at the point $x=2$.

(ii) At what point is the gradient equal to 1 ?

4. y is known to be a function of x of the type $mx+c$, m and c being constants. If $y=43$ when $x=17$, and $y=34$ when $x=14$, express y as a function of x.

5. A flexible rod AB is built into a wall at A and supported at B; its length is l in. and the sag x in. from A is y in., where

$$y=\frac{1}{10l^3}x^2(l-x)(3l-2x).$$

Find the value of x for which the sag is greatest.

K. 3.

1. Find $\dfrac{dy}{dx}$ (i) when $y=4x^2-3+\dfrac{1}{x}$,

(ii) when $y=16x^3-4x^2+2x-5$,

(iii) when $y=5x^{\frac{1}{3}}+x^{-\frac{1}{3}}$.

2. Find the gradient at $x=a$ of the graph

$$y=x^3-6x^2-15x+5.$$

Find for what values of x the gradient of this graph is zero. Sketch the graph roughly.

3. The distance, s feet, of a particle moving in a straight line from a fixed origin is given by $s=4+6t+3t^2$, where t is the number of seconds for which it has been moving. Find its velocity, (i) 2 seconds, (ii) 10 seconds after it started.

4. A particle moves in a straight line in such a way that its velocity after any time t varies inversely as its distance x from the origin. Write down a differential equation connecting $\dfrac{dx}{dt}$ and x.

5. If $f(x) \equiv x^3 - 5x^2 + x + 7$ and if $f(y+a)$ when expanded contains no term in y^2, find a.

K. 4.

1. Find the value of (i) $\underset{h\to 0}{\text{Lt}} \left\{ \dfrac{\dfrac{1}{(x+h)^2-2} - \dfrac{1}{x^2-2}}{h} \right\}$.

(ii) $\underset{n\to\infty}{\text{Lt}} \dfrac{n^2+1}{2n^2+1}$.

2. Differentiate the following with respect to x:

(i) $\sqrt{6x}$. (ii) $6\sqrt{x}$. (iii) $\dfrac{4}{x^3}$. (iv) $x^2 + \dfrac{1}{x^2}$.

3. For what values of x is the function $x^3 - 12x$ equal to zero? Find its derived function and find for what values of x the derived function is zero. From these data sketch a rough graph of the function.

4. There are 15 cubic feet of water in a bath. The water is allowed to run out, and after t seconds θ cubic feet of water have run out. If the rate at which the water runs out is proportional to the quantity of water left in the bath, find a differential equation connecting $\dfrac{d\theta}{dt}$ and θ.

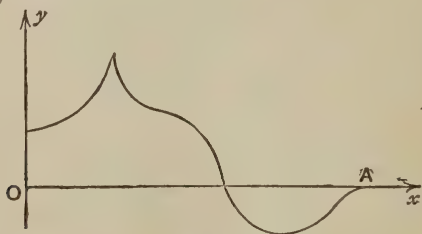

FIG. 51.

5. Fig. 51 represents the graph of $y = f(x)$, unit 1 cm. on each axis; describe in general terms the variation in the values of (i) y; (ii) $\dfrac{dy}{dx}$; (iii) $\dfrac{d^2y}{dx^2}$ as x varies from 0 to OA.

K. 5.

1. (i) Describe the changes of sign in the function $\dfrac{3(x-2)(x-4)}{(x-1)(x-5)}$ when x increases from -1 to 6.

(ii) Find correct to one significant figure its value when $x=1{\cdot}0001$.

(iii) Can you find a value of x for which the function equals 3 ?

(iv) Find by logarithms the value of the function when $x=1000$.

(v) Simplify $\dfrac{3(x-2)(x-4)}{(x-1)(x-5)}-3$, and evaluate

$$\operatorname*{Lt}_{x\to\infty}\frac{3(x-2)(x-4)}{(x-1)(x-5)}.$$

(vi) Sketch the graph of the function for positive values of x.

2. Differentiate the following functions of t with respect to t:

(i) $4t+\dfrac{4}{t}$. (ii) $3t^3-4t^{-1}+5t^{-3}$. (iii) $5t^{\frac{5}{2}}$.

3. Differentiate the function x^3-27x. For what values of x is its rate of change zero ? What are the maximum and minimum values of this function ? Sketch its graph.

4. The weight of a certain solid is given by the formula

$$10x^2(15-2x),$$

where x is variable. Show that as x increases from 0 to 5 the solid gets steadily heavier. What happens after that ?

5. If $f(x)=x^2+3x$, find the value of

$$\text{(i)}\ \frac{f(x+h)-f(x)}{h}.$$

$$\text{(ii)}\ \frac{f(x+h)-f(x)-\{f(x)-f(x-h)\}}{h^2}.$$

Find the limits to which these two expressions tend, as $h\to 0$.

K. 6.

1. (i) Is $\dfrac{x^2-1}{x-1}$ always equal to $x+1$?

(ii) Can you find a value of x for which $\dfrac{x^2-1}{x-1}$ equals 2 ?

(iii) Find the value of $\operatorname*{Lt}_{x\to 1}\dfrac{x^2-1}{x-1}$.

2. Find the turning points of the function x^3+3x^2-9x+6, and determine whether they are maxima or minima.

3. Show that the function x^3+5x-6 steadily increases as x increases from $-\infty$ to $+\infty$.

One root of the equation $x^3+5x-6=0$ is $x=1$. Show why it cannot have another real root.

4. A piece of wire 8 inches long is bent so as to form a rectangle. If one side of the rectangle is x'', express the area as a function of x, and find the value of x for which the area is a maximum.

5. The distance, s feet, that a stone falls is proportional to the square of the time, t seconds, for which the stone has been falling. Express this fact as an equation connecting s and t, and prove that the velocity of the stone varies directly as the time for which it has been falling.

K. 7.

1. Differentiate the function $5-3x-2x^3$ and show that the function steadily decreases as x varies from $-\infty$ to $+\infty$, and that it can only be zero, when $x=1$. Draw roughly the graph of this function.

2. A particle moves along the straight line OA. B is a point on OA such that $OB=5$ feet, and the particle starts from B and moves away from O in such a way that its velocity is proportional to its distance from O. If its initial velocity is 10 feet per second, find an equation connecting $\frac{dx}{dt}$ and x, where x feet is its distance from the starting point B, and t seconds is the time for which it has been moving.

3. The volume of a cone is $\frac{1}{3}\pi r^2 h$ and its curved surface is πrl, where r is the radius, h the vertical height and l $(=\sqrt{h^2+r^2})$ the slant height of the cone. A conical tent is to be made to hold 400 cubic feet of air. Express its height as a function of r, and then express its surface S as a function of r. Find the value of S, if $r=7$. For what value of r is S least?

4. Draw a rough sketch of the graph of the function $18x^2-x^4$, and find its maximum and minimum values. (Certificate.)

5. If $y=2x^2$ and $x=3z+1$, express (i) δy in terms of δx; (ii) δx in terms of δz; (iii) δy in terms of δz. Hence find $\frac{dy}{dz}$.

K. 8.

1. Tabulate the values of the function $x+\frac{4}{x}$, as x varies from -4 to $+4$. Draw an accurate graph of this function, and read off the values of x for which this function is a maximum or minimum.

Verify your answers by differentiating the function, and calculating the values of x required.

2. Draw a tangent to your graph (Question 1) at the point $x=3$, and estimate its gradient. Check your answer by calculating the value of $\frac{dy}{dx}$ when $x=3$.

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

(i) when $y=3x^3-7x$; (ii) when $y=5x^2-4$;

(iii) when $y=7x+6$; (iv) when $y=5$;

(v) when $y=\frac{6}{x}$.

4. Draw a rough sketch of the graph of the function $75x-4x^3$, and find its maximum and minimum values. (Certificate.)

5. A rectangular plate 20 inches long and $3x$ inches wide has a circular hole, radius x inches, cut out of it. Express the area of the remaining portion as a function of x, and find the value of x for which the area remaining is a maximum.

K. 9.

1. If $y=ax^2+bx$, prove that $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$.

2. The sides of a triangle are 5, 5, $2x$ cm.; its area is A sq. cm. Express A as a function of x; and find the maximum area of the triangle. (C.S.C.)

3. Ox, Oy are horizontal and vertical lines: the graph of $y=x^2$ represents a hill side on the scale, unit for x-axis$=100$ yards, unit for y-axis$=1$ foot. What is the average slope of the hill, (i) from $x=1$ to $x=2$; (ii) from $x=1$ to $x=1\cdot1$; (iii) from $x=1$ to $x=1+h$? What is the gradient of the hill at $x=1$?

4. A, B, C are three points on Ox ; APB is a semicircle above Ox and BQC a semicircle below OX ; the two semicircles form part of the graph of $y=f(x)$. Describe the changes of sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ between A and C.

5. An open cardboard box with square ends is fitted with an overlapping lid which covers the open top, the whole of two square ends and one other side. The total area of the cardboard is 8 sq. feet. What is the maximum volume of the box ?

K. 10.

1. The electromotive force of a certain type of cell has been found to vary with the temperature as follows :

Temperature (Centigrade)	15	20	25	30
E.M.F. in volts - - -	1·4340	1·4284	1·4233	1·4188

What is the average fall in E.M.F. per degree of temperature, when the temperature rises from (i) 15° C. to 30° C. ; (ii) 15° C. to 25° C. ; (iii) 15° C. to 20° C. ? Illustrate the table by a graph, and by drawing a tangent estimate the rate of fall of E.M.F. at 20° C.

2. Find $\frac{dy}{dx}$ if (i) $y=\frac{x^3-1}{x}$; (ii) $y^3=2x^2$.

3. If a steamer travels at v knots, the cost of a certain journey is $8\left(\frac{1500}{v}+0\cdot4v^2\right)$£. What is the most economical speed ?

4. For what values of x is the function $108x-x^4$, (i) positive, (ii) an increasing function, (iii) a maximum ? Sketch its graph.

5. A closed vessel which tapers to a point at its top and bottom is such that when the depth of liquid in it is x feet, the volume of the liquid is $12x^2-x^3$ cu. feet. Find (i) its internal height ; (ii) the area of its cross-section halfway up ; (iii) the total volume.

K. 11.

1. A bullet fired vertically upwards rises s feet in t seconds, where $s=500t-16t^2$. (i) Find its height after 10 sec., 11 sec., 10·1 sec. ; (ii) What is its average speed in the intervals 10 to 11 sec. and 10 to 10·1 sec. ? (iii) What is its average speed in the interval 10 to $10+h$ sec. ? (iv) What is its velocity after 10 sec. ?

2. Use logarithm tables to evaluate $x \times \delta(\log x) \div \delta x$, when (i) $x=5$, $\delta x=0\cdot 1$; (ii) $x=7$, $\delta x=0\cdot 1$; (iii) $x=9$, $\delta x=0\cdot 1$. What do you infer from these results ?

3. Draw freehand the graph of a function $y=f(x)$ for which

(i) over a portion AB $\dfrac{dy}{dx}$ is positive and $\dfrac{d^2y}{dx^2}$ is negative ;

(ii) over a portion CD $\dfrac{dy}{dx}$ is negative and $\dfrac{d^2y}{dx^2}$ is positive.

4. A circular electric current of radius 1 foot exerts a force P on a small magnet placed with its axis perpendicular to the plane of the wire and at a distance x feet from the centre, where P varies as $\dfrac{x}{(1+x^2)^{\frac{5}{2}}}$. For what value of x is P greatest ?

5. ABP is a triangle, right-angled at B, having $AB > BP$; $AB=12''$; the perpendicular bisector of AP cuts AB at Q. Find the length of AQ if triangle PBQ is of maximum area.

K. 12.

1. If $xy=5$, find

(i) $\dfrac{dy}{dx}$ in terms of x; (ii) $\dfrac{dx}{dy}$ in terms of x; (iii) $\dfrac{dy}{dx} \times \dfrac{dx}{dy}$.

2. What is the greatest value of $3x-5x^2y$ if $xy^2=4$ and y is positive ?

3. If $f(x)=x^2-3x+5$, (i) express $\dfrac{f(x+h)-f(x)}{h}$ in terms of x, h; (ii) find the limit of this expression when $h \to 0$; (iii) for what value of x is this limit zero ; (iv) interpret the results of (i), (ii), (iii) geometrically.

4. If $y=\dfrac{1}{z}$ and $z=1-x^2$, (i) express δy in terms of z, δz; (ii) express δz in terms of x, δx; (iii) express δy in terms of x, δx; (iv) find $\dfrac{d}{dx}\left(\dfrac{1}{1-x^2}\right)$.

5. The function $y=3x+\dfrac{a}{x}$ decreases as x increases from 0 to 5, and increases as x increases from 5 to 10. (i) What is the value of a ? (ii) what is the value of $\dfrac{d^2y}{dx^2}$ when $x=5$ and $x=-5$? (iii) find maximum and minimum values of the function, distinguishing between them ; (iv) explain your results by a sketch of the function.

CHAPTER IV.

INTEGRATION.

Example I. If $\dfrac{dy}{dx}=x^2+7$, express y in terms of x.

We know that $$\frac{d}{dx}x^3=3x^2;$$

$$\therefore \frac{d}{dx}(\tfrac{1}{3}x^3)=x^2.$$

Also $$\frac{d}{dx}(7x)=7;$$

$$\therefore \frac{d}{dx}(\tfrac{1}{3}x^3+7x)=x^2+7.$$

$\therefore$ *one* value of y is given by $y=\frac{1}{3}x^3+7x$.

But $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x-99) \text{ is also } x^2+7,$$

or $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x+2\tfrac{1}{4}) \text{ is also } x^2+7,$$

because $\dfrac{d}{dx}$ (any constant) is 0.

$\therefore$ $y=\frac{1}{3}x^3+7x+c$, where c is any constant, satisfies the given equation, and is called its *general solution.*

Note.—An equation involving $\dfrac{dy}{dx}$ or $\dfrac{d^2y}{dx^2}$, etc., is called a *differential equation*: and the process of expressing y as a function of x is called *solving* or *integrating* the differential equation.

Whenever a differential equation is integrated, an arbitrary constant enters into the result.

EXERCISE IV. a.

1. Prove that $y = x^4 - 5$ is *a* solution of the differential equation $\dfrac{dy}{dx} = 4x^3$. What is the general solution ?

2. Prove that $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + 3$ is *a* solution of the differential equation $\dfrac{dy}{dx} = x^4 + x$. What is the general solution ?

3. The graph of a curve $y = f(x)$ is such that its gradient is everywhere equal to $2x$. (i) What is the differential equation for the curve ? (ii) Prove that $y = x^2$, $y = x^2 + 2$, $y = x^2 - 1$ are each possible solutions, and interpret this result geometrically by drawing the graphs.

4. Prove that $y = \frac{1}{12}x^4 + 97x - 13$ is *a* solution of the differential equation $\dfrac{d^2y}{dx^2} = x^2$. What is the general solution ?

5. Solve $\dfrac{dy}{dx} = x$.

6. Solve $\dfrac{dy}{dx} = 3x^2 - 2x$.

7. Solve $\dfrac{dy}{dx} = 4$.

8. Solve $\dfrac{d^2y}{dx^2} = 0$.

9. (i) What is $\dfrac{d}{dx}x^5$? (ii) Solve $\dfrac{dy}{dx} = x^4$.

(iii) Solve $\dfrac{dy}{dx} = 3x^4$.

10. (i) What is $\dfrac{d}{dx}x^8$? (ii) Solve $\dfrac{dy}{dx} = x^7$.

(iii) Solve $\dfrac{dy}{dx} = 5x^7 - 2$.

11. (i) What is $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$? (ii) Solve $\dfrac{dy}{dx} = \dfrac{1}{x^2}$.

(iii) Solve $\dfrac{dy}{dx} = \dfrac{4}{x^2}$.

12. (i) What is $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right)$? (ii) Solve $\dfrac{dy}{dx} = \dfrac{1}{x^5}$.

(iii) Solve $\dfrac{dy}{dx} = 1 - \dfrac{3}{x^5}$.

13. (i) What is $\dfrac{d}{dx}(\sqrt{x})$? (ii) Solve $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$.

(iii) Solve $\dfrac{dy}{dx} = x + \dfrac{3}{\sqrt{x}}$.

14. Solve $\frac{dy}{dx}=5x^2-3x+9$.

15. Solve $\frac{dy}{dx}=x^5-x^2+3$.

16. Solve $\frac{dy}{dx}=x^3+\frac{1}{x^3}$.

17. Solve $\frac{dy}{dx}=5\sqrt{x}$.

18. Solve $x^3\frac{dy}{dx}=3x+5$.

19. Solve $\frac{dy}{dx}=(x-2)(x-3)$.

20. Solve $\frac{d^2y}{dx^2}=x^2-x$.

21. Solve $\frac{dy}{dx}=3x^{100}+\frac{5}{x^{100}}$.

22. If $\frac{dy}{dx}=x-2$, prove that $y=\frac{1}{2}x^2-2x+c$, and find c if $y=5$ when $x=6$.

23. If $\frac{ds}{dt}=50-32t$, and if $s=48$ when $t=2$, find s in terms of t.

24. If $\frac{d^2s}{dt^2}=4t$, and if, when $t=0$, $\frac{ds}{dt}=5$ and $s=8$, express s in terms of t.

25. If a body travels s feet in t seconds, its velocity after t seconds is equal to $\frac{ds}{dt}$. Given that $\frac{ds}{dt}=3+4t$, and that it passed a point A after 5 seconds from the start, find its distance from A after another 5 seconds.

26. The gradient of the graph of the function in Fig. 52 is given by $\frac{dy}{dx}=3x^2-1$, and the graph cuts Oy at a distance 3 units from O. Find the function.

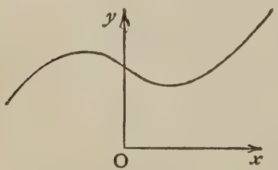

FIG. 52.

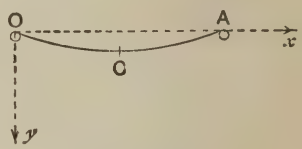

FIG. 53.

27. Fig. 54 represents a uniform lathe OA 20 feet long supported at its ends, which are at the same level. Bending slightly under its own weight, its form obeys the law $\frac{d^2y}{dx^2}=0{\cdot}0001\,(x^2-20x)$; unit 1 foot on each axis.

(i) State from first principles the value of $\frac{dy}{dx}$ at the mid-point C where $x=10$.
(ii) What is the value of y when $x=0$?
(iii) Find y in terms of x, and then put $x=20$ in the result; what should be the value of y ?
(iv) What is the slope of the lathe at each of its ends ?
(v) What is the sag of the lathe at its mid-point ?

28. A chain OA, 100 feet long, hangs vertically from O. Its weight per unit length steadily diminishes from O to A, and at any point x feet from O is $(4-0\cdot03x)$ lb. per foot. If the work done in winding it up on a drum at O is W foot-lb., then $\frac{dW}{dx}=4x-0\cdot03x^2$; find the work done in winding it up.

29. A stone is thrown horizontally with a velocity of 12 feet per sec. from the top of a tower O, 100 feet high; Ox is horizontal and Oy vertical (see Fig. 54). If P is its position t seconds after starting, and if $ON=x$ feet, $PN=y$ feet, then x, y obey the laws $\frac{d^2x}{dt^2}=0$; $\frac{d^2y}{dt^2}=32$, neglecting air resistance.

(i) Express x and y each in terms of t.

(ii) Express y in terms of x.

(iii) How long does it take to reach the ground?

(iv) How far from the foot of the tower does it strike the ground?

(v) What is the gradient of the path of the stone where it strikes the ground?

(vi) A man is standing on the ground 50 feet from the foot of the tower; where is the stone when it appears to him to be coming straight at him?

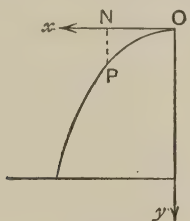

FIG. 54.

30. An elastic spiral spring attached to A, and hanging vertically, supports a body at O. The body is pulled down 3″ and let go. It now oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x in., v and x obey the law $\frac{d}{dx}(\tfrac{1}{2}v^2)=-\tfrac{1}{16}x$.

(i) Find its velocity when it passes O.

(ii) How far must it be pulled down in order to have a velocity of 2 in. per sec. when passing O?

31. The rate at which a body cools is proportional to the excess of its temperature above that of the atmosphere. The temperature of a body is $T°$ C. at x minutes past one in a room

of temperature $K°$ C. Express the law of cooling by a differential equation.

32. In climbing a mountain, the rate at which the atmospheric pressure p lb. per sq. in. decreases per unit increase of the height x feet is proportional to the pressure. Express this law by a differential equation.

Notation. The relation $\frac{dy}{dx}=f(x)$ is usually expressed in the form $y=\int f(x)\,dx$.

In other words, $\int f(x)\,dx$ represents the function which when differentiated with respect to x gives $f(x)$.

For example
$$\int x^3\,dx \equiv \frac{x^4}{4}+c,$$
where c is a constant, because
$$\frac{d}{dx}\left(\frac{x^4}{4}+c\right)\equiv x^3.$$

AREAS AND VOLUMES.

Example II. Figure 55 represents the graph of $y=\frac{x^2}{5}+1$, unit $\frac{1}{2}''$ on each axis.

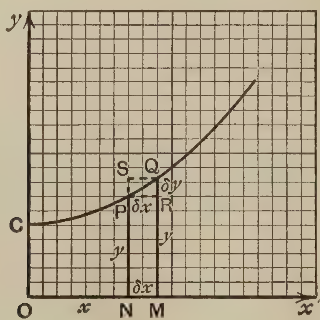

FIG. 55.

If $ON=x$ in., and if the area bounded by $CO.\ ON$, NP and arc CP is A sq. in., express A in terms of x.

With the usual notation

δA represents the area bounded by PN, NM, MQ and arc PQ;

$$\therefore\ \delta A > \text{rectangle } PNMR;$$

$$\therefore\ \delta A > y\,\delta x \quad \text{or} \quad \frac{\delta A}{\delta x} > y,$$

and δA < rectangle $SNMQ$;

$$\therefore\ \delta A < (y+\delta y)\,\delta x \quad \text{or} \quad \frac{\delta A}{\delta x} < y+\delta y;$$

$$\therefore\ y+\delta y > \frac{\delta A}{\delta x} > y;$$

$$\therefore \text{ when } \delta y \to 0,\ \frac{dA}{dx} = y = \frac{x^2}{5}+1;$$

$$\therefore\ A = \frac{x^3}{15}+x+c, \text{ where } c \text{ is constant.}$$

But when $x=0$, the area $A=0$;

$$\therefore\ c=0;$$

$$\therefore\ A = \frac{x^3}{15}+x.$$

Would there be any difference in the above argument if the graph sloped downwards instead of upwards?

Example III. Figure 56 represents the graph of $y=\dfrac{x^2}{5}+1$, not drawn to scale; if $OH=2$, $OK=7$, find the area bounded by AH, HK, KB and the arc AB.

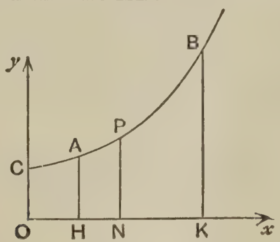

FIG. 56.

It is proved above in Example II. that if $ON=x$, the area of $CONP$ is A, where $\dfrac{dA}{dx}=\dfrac{x^2}{5}+1$;

$$\therefore\ A = \int\left(\frac{x^2}{5}+1\right)dx$$

$$= \frac{x^3}{15}+x+c, \text{ where } c \text{ is a constant;}$$

$\therefore$ the area of $COKB = \dfrac{7^3}{15} + 7 + c$,

and the area of $COHA = \dfrac{2^3}{15} + 2 + c$;

$$\therefore \text{ the area of } AHKB = \left[\frac{7^3}{15} + 7\right] - \left[\frac{2^3}{15} + 2\right]$$

$$= \frac{343 - 8}{15} + 5 = \frac{335}{15} + 5 = \frac{67}{3} + 5 = 27\tfrac{1}{3}.$$

Note the disappearance of the arbitrary constant c, owing to the subtraction.

Notation. The symbol $\int_a^b f(x)\,dx$ is used to express the value of $\int f(x)\,dx$ when $x = b$ *minus* the value of $\int f(x)\,dx$ when $x = a$.

Thus

$$\int_2^7 x^2 dx \equiv \left[\frac{x^3}{3}\right]_{x=7} - \left[\frac{x^3}{3}\right]_{x=2} = \frac{7^3}{3} - \frac{2^3}{3} = \frac{343 - 8}{3} = \frac{335}{3},$$

and we often write $\displaystyle\int_2^7 x^2\,dx \equiv \left[\frac{x^3}{3}\right]_2^7.$

Looking back at Example III. we see that the area of $AHKB$ can be written in any of the following forms:

$$\int_2^7 y\,dx \equiv \int_2^7 \left(\frac{x^2}{5} + 1\right) dx \equiv \left[\frac{x^3}{15} + x\right]_2^7.$$

The expression $\int f(x)\,dx$ is called an *indefinite* integral because its value contains an arbitrary constant.

The expression $\int_a^b f(x)\,dx$ is called a *definite* integral because the arbitrary constant of integration disappears automatically.

Examples II., III. show that $\int_a^b f(x)\,dx$ is the area bounded by two ordinates $x = a$, $x = b$, the x-axis and an arc of the graph of $y = f(x)$.

EXERCISE IV. b.

1. Fig. 57 represents the graph of $y=x^2$; $ON=x$; (i) find in terms of x the area bounded by ON, NP, arc OP; (ii) find the area $AHKB$, where $OH=1$, $OK=2$; (iii) express this area as a definite integral.

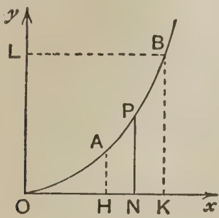

FIG. 57.

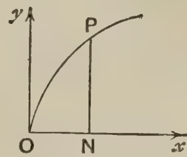

FIG. 58.

2. Fig. 58 represents the graph of $y^2=x$; $ON=x$; find the area ONP.

3. Interpret geometrically $\int_1^3 x\,dx$ and evaluate it (i) by direct calculation, (ii) geometrically.

4. Fig. 59 represents part of the graph of $y=(x-1)(4-x)$; (i) find OA and OB; (ii) find the area between Ox and the portion of the curve above Ox.

FIG. 59.

5. Draw the graph of $\frac{1}{x^2}$ and find the area (i) between it and the ordinates $x=1$, $x=2$ and the x-axis; (ii) between it and the lines $y=1$, $y=2$ and the y-axis.

6. If Fig. 57 represents the graph of $y=x^3$; prove that the area BLO is three times the area BKO.

7. Find the values of

(i) $\int_1^2 (x^2+5)\,dx$; (ii) $\int_2^4 \frac{dx}{x^3}$;

(iii) $\int_{-1}^{+1} (x-x^2)\,dx$; (iv) $\int_0^{\frac{1}{2}} (1+x)^2\,dx$.

8. Fig. 60 represents a solid formed by rotating part of the graph of $y=x^2$ about Ox.

$ON=x$, $NP=y$, $OM=x+\delta x$, $QM=y+\delta y$, $OH=\frac{1}{4}$, $OK=\frac{3}{4}$.

The figure shows cross-sections perpendicular to Ox.

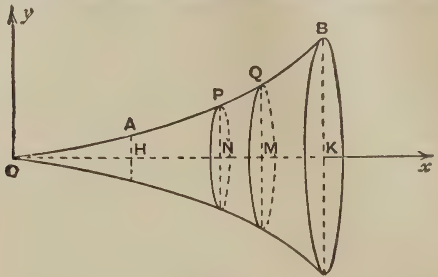

FIG. 60.

V is the volume of the solid between O and the cross-section through P.

(i) What is the area of the cross-section through P ?

(ii) What is the area of the cross-section through Q ?

(iii) Interpret $\pi y^2\,\delta x$ geometrically.

(iv) Prove that $\pi(y+\delta y)^2\,.\,\delta x > \delta V > \pi y^2\,.\,\delta x$.

(v) Prove that $V=\pi\int_0^x y^2\,dx$.

(vi) Express V in terms of x.

(vii) Find the volume between the cross-sections at A and B, and express it also as a definite integral.

(viii) If the graph of $y=f(x)$ is drawn, interpret geometrically $\int_0^x \pi y^2\,dx$.

(ix) If the graph of $y=f(x)$ is drawn, interpret geometrically $\int_0^y \pi x^2\,dy$.

9. (i) Draw the graph of $y=\frac{1}{2}x$; take a point P on it and draw PN perpendicular to Ox.

(ii) What solid is formed by revolving OP about Ox ?

(iii) Find, by the method of Ex. 8, the volume of the solid between O and the section through P perpendicular to Ox, in terms of x.

10. The graph of $y=\dfrac{1}{x}$ is rotated about Oy; find the volume of the solid between two planes perpendicular to Oy and at distances 1 and 2 from O.

11. Fig. 61 represents a section of a hemisphere of radius 10 in., centre O; $ON = x$ in.

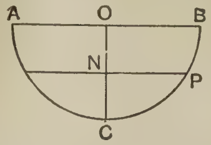

FIG. 61.

(i) Find PN in terms of x.

(ii) Find the area of the cross-section of the hemisphere at N perpendicular to ON.

(iii) Find an approximate expression for the volume of the hemisphere contained between two parallel planes at distances x and $x + \delta x$ from O.

(iv) If the volume between the sections through O and through N, perpendicular to ON, is V cu. in., express V in terms of x.

(v) Express as a definite integral the volume of the spherical segment between C and the section through N; and simplify it.

(vi) Find the volume of the hemisphere.

12. A basin is formed by revolving the graph of $y = \frac{1}{8}x^4$ about the vertical axis Oy; how much water is in the basin when the greatest depth is 4 inches? [Unit for each axis, one inch.]

13. (i) Sketch the graph of $y = \frac{1}{27}(2x+3)(9-2x)$ from $x = 0$ to $x = 3$.

(ii) A beer barrel is formed by rotating this portion of the graph about Ox. [Unit for each axis, one foot.] Find the capacity of the barrel in cu. feet.

14. (i) Sketch the graphs of $y^2 = x$ and $x^2 = 8y$ for positive values of x.

(ii) Show that they cross each other at the point (4, 2).

(iii) Show that the area of the portion common to both is

$$\int_0^4 \left(\sqrt{x} - \frac{x^2}{8}\right) dx, \text{ and evaluate it.}$$

15. If $f(x) = \int_0^x \left(1 - \frac{y^2}{4}\right) dy$, find the value of $\int_0^1 f(x)\, dx$. [This may be stated more shortly as follows: find the value of

$$\int_0^1 \int_0^x \left(1 - \frac{y^2}{4}\right) dy\, dx.]$$

16. Using the notation explained in Ex. 15, find the value of

$$\int_{-1}^{1}\int_{1}^{x} y(1-y)\,dy\,dx.$$

17. Show that $\int_0^1 (x+2)(x+3)\,dx = \int_1^2 (x+1)(x+2)\,dx$. Interpret this result geometrically.

18. A lathe of length 10 feet is built into a wall horizontally at one end and is deflected by a weight fastened at the other end. The deflection at x feet from the wall is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_0^v 0{\cdot}0005\,(10-u)\,du$; [or more shortly,

$$\int_0^x\int_0^v 0{\cdot}0005(10-u)\,du\,dv].$$

Find the deflection (i) when $x=10$, (ii) when $x=5$.

19. A lathe of length 10 feet is supported at its ends and loaded at the middle. The deflection at x feet from one end is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_v^5 0{\cdot}0005u(10-u)\,du$. Find the deflection at the mid-point.

20. The moment of inertia of a cylinder, radius r in., height h in., about its axis is $2\pi\rho h\int_0^r x^3\,.\,dx$, where ρ lb. is the mass per cu. inch. If the mass of the cylinder is W lb., express the moment of inertia in terms of r, W.

21. The moment of inertia of a sphere, radius r in., about a diameter is $\pi\rho\int_0^r (r^2-x^2)^2dx$, where ρ lb. is the mass per cu. inch. If the mass of the sphere is W lb., express the moment of inertia in terms of r, W.

22. A plane area is enclosed between the curve $y^2=2x^3$ and the line $x=3$. The distance of the centre of gravity of this area measured along the axis of x from the origin is given by the formula $S=\int_0^3 yx\,dx \div \int_0^3 y\,.\,dx$. Evaluate this expression.

23. A solid is formed by the rotation of the portion of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ lying between the lines $x=0$ and $x=5$ about the axis of x. The distance of the centre of gravity of this solid measured from the origin along its central axis is given by the formula $S=\int_0^5 \pi y^2x\,dx \div \int_0^5 \pi y^2\,.\,dx$. Evaluate this expression.

24. A gas is compressed in a cylinder by a piston. If the face of the piston is initially a inches from the inside end of the cylinder, and if the piston travels forward until this distance is b inches, the work done is $-\int_a^b \frac{660}{x^{1.4}}\,dx$ ft.-lb. If $a=10$, find the work done in compressing the gas to half its volume.

25. A new cure for chilblains was tried on 8 people. It succeeded in 5 cases and failed in 3 cases. The chance that it succeeds in the next two cases is

$$\int_0^1 x^7(1-x)^3\,dx \div \int_0^1 x^5(1-x)^3\,dx.$$

Find what the chance is.

26. Two points are taken at random on a line 12 inches long. The chance that the distance between them does not exceed 3 inches is $\int_3^{12}(x-3)\,dx \div \int_0^{12} x\,dx$. Evaluate this chance.

27. The portion of the parabola $y=\frac{1}{4}x^2$ (the y-axis being drawn vertically downwards) above $y=5$ is taken and submersed in a liquid in a vertical plane with its vertex uppermost and 1 unit distance below the surface. Then the depth of the centre of pressure below the surface is

$$\int_0^5 (1+x)^2\sqrt{x}\,dx \div \int_0^5 (1+x)\sqrt{x}\,dx.$$

Evaluate this expression.

28. The density of a spherical body of radius a feet varies directly as its distance from the centre. If ρ is the density (*i.e.* the mass in lb. per cu. ft.) at the surface, (i) prove that the mass of the spherical shell whose inner and outer radii are x and $x+\delta x$ feet, is approximately $\frac{4\pi\rho x^3}{a}\,.\,\delta x$ lb.; (ii) find the total mass of the body.

29. The density of a chain AB varies as its distance from A (*i.e.* at a point x feet from A, the material weighs kx lb. per foot length); its length is 20 feet; the lighter half weighs 30 lb. Find the weight of the heavier half.

30. The velocity of a body after t sec. from its start is v feet per sec., where v is a given function of t, viz. $v=f(t)$; (i) what meanings can you give to $\int v\,dt$? (ii) Given the time-velocity graph of a moving particle, how could you find the distance travelled in a given time?

31. Fig. 62 represents a mound with the rectangle $ABCD$ as base and with its sides sloping up to the ridge EF, which is

situated symmetrically. $AB=30'$, $AD=18'$, $EF=10'$; and the height of EF above the base is 12 feet.

(i) Prove that the area of the section of the mound parallel to the base and h feet below EF is

$$\tfrac{5}{3}h\,(h+6) \text{ sq. ft.}$$

(ii) Find the volume of the mound.

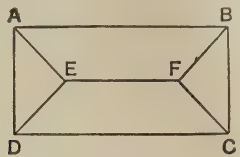

FIG. 62.

32. (i) Sketch the half of the ellipse $\dfrac{x^2}{4}+y^2=81$ which lies between $x=0$ and $x=18$.

(ii) This portion is rotated about Ox to form the explosive head of a torpedo; unit on each axis, 1 inch. Find the volume of the explosive in the head.

SIMPSON'S RULE FOR APPROXIMATION OF AREAS.

PA, QB are the perpendiculars from P, Q to a line AB.

AB is divided into any even number, say $2n$, equal parts, and the ordinates $y_1, y_2, y_3, \dots y_{2n+1}$ are drawn from each point of division and from the beginning and end of the line to the curve.

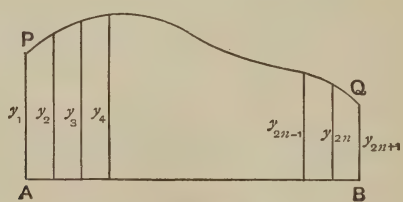

FIG. 63.

Let $$h=\frac{AB}{2n}.$$

Then the area contained by PA, AB, BQ and the arc PQ is

$$\simeq \tfrac{1}{3}h\,[y_1+y_{2n+1}+2(y_3+y_5+y_7+\dots+y_{2n-1}) \\ +4(y_2+y_4+y_6+\dots+y_{2n})].$$

Evaluation of Definite Integrals. Simpson's rule may be used to evaluate definite integrals, because, as we have seen (p. 74), a definite integral may be regarded as representing an area.

Example IV. Find the approximate value of $\int_1^{10} \frac{1}{x} dx$.

It is necessary to divide the interval $x=1$ to $x=10$ into an *even* number of strips. Take 12 strips so that the breadth of each strip is $\frac{10-1}{12}=\frac{3}{4}$; we must therefore find the values of $\frac{1}{x}$ for $x=1$, $1\frac{3}{4}$, $2\frac{1}{2}$, $3\frac{1}{4}$, 4, $4\frac{3}{4}$, $5\frac{1}{2}$, $6\frac{1}{4}$, 7, $7\frac{3}{4}$, $8\frac{1}{2}$, $9\frac{1}{4}$, 10. Time may be saved by using a table of Reciprocals.

The work should always be arranged according to the following scheme:

	Values of $\frac{1}{x}$.		
		Even ordinates.	Other odd ordinates.
1	1		
1·75		·5714	
2·5			·4000
3·25		·3077	
4			·2500
4·75		·2105	
5·5			·1818
6·25		·1600	
7			·1429
7·75		·1290	
8·5			·1176
9·25		·1081	
10	·1		
	1·1	1·4867	1·0923
		4	2
		5·9468	2·1846

1·1
5·9468
2·1846
9·2314

$$\therefore \int_1^{10} \frac{1}{x} dx \simeq \tfrac{1}{3} \times \tfrac{3}{4} \times 9{\cdot}2314 \simeq 2{\cdot}308.$$

Note. Correct to 3 places of decimals, the actual value is 2·303. This would have been obtained if we had taken 20 instead of 12 strips.

EXERCISE IV. c.

1. (i) Write down the values of x^2 for $x=0 \cdot 1$, $0 \cdot 2, \ldots, 0 \cdot 9$, and then use Simpson's rule to find the area bounded by the x-axis, the graph of $y=x^2$ and the ordinate through $x=1$.

(ii) Evaluate this area by calculation.

2. (i) Draw the graph corresponding to the following table of values :

$x=0$	1	2	3	4	5	6	7	8	9	10
$y=0$	7	8	11	15	18	21	18	14	6	0

(ii) Find its area by " counting squares."
(iii) Find its area by " Simpson's rule."

3. Draw a semicircle, diameter 10 cm., and find its area
(i) by Simpson's rule (ten strips),
(ii) by formula.
Give the approximate error per cent. of (i).

4. Find the area bounded by the graph of $y=\frac{60}{x}$, the x-axis, and the ordinates $x=5$ and $x=10$, by Simpson's rule.
Assuming that the area is $41 \cdot 6$, find the approximate error per cent.

5. The velocity of a train after leaving a station is shown in the following table :

t, time in minutes -	0	2	4	6	8	10	12	14	16	18	20
v speed in yards per minute -	0	50	110	160	230	290	360	410	470	530	570

Find in miles the distance travelled in the first 20 minutes,

6. In finding the volume of earth in a cutting between two sections A and B, the following measurements were obtained :

	Distance from A in yards.	Area of section of cutting in sq. feet.
At A - -	0	2000
	8	3000
	12	3420
	19	3900
	24	3900
	30	3700

(i) Represent these results graphically.

(ii) Find the volume of earth in the cutting in cu. ft. by Simpson's rule. (C.S.C.)

7. The speed of a car, v miles per hour, was noted at intervals of 12 seconds over a period of 2 minutes.

Time in seconds =	0	12	24	36	48	60	72	84	96	108	120
$v=$	5	3	2·5	3	5·7	10·3	14	10·5	5·6	2	0

Find the distance travelled in feet. (C.S.C.)

8. Use Simpson's rule to find an approximate result for the volume of a hemisphere, radius 10 cm., taking sections parallel to the base of the hemisphere. Compare this with the result given by the formula $\frac{2}{3}\pi r^3$ for a hemisphere.

9. (i) Sketch the graph of $y=\log x$ from $x=1$ to $x=10$.

(ii) Interpret geometrically $\int_1^2 \log x \,.\, dx$.

(iii) Use Simpson's rule to find an approximate value of this definite integral.

10. By using Simpson's rule, find an approximate value of $\int_0^{10} \sqrt{1+x^3}\, dx$.

REVISION PAPERS. L. 1–10.

L. 1.

1. Find correct to two significant figures the value of x for which $\sqrt{x}-x^{1 \cdot 06}$ is a maximum.

2. (i) Solve $\dfrac{dy}{dx}=x-\sqrt{x}$, given $y=1$ when $x=1$.

(ii) Solve $\dfrac{d^2y}{dx^2}=\dfrac{1}{x^3}$, given $y=4$ when $x=\frac{1}{2}$ and $y=-2$ when $x=-\frac{1}{2}$.

3. Find the values of (i) $\int\left(5x^2+\dfrac{4}{x^2}\right)dx$; (ii) $\int\left(x^{1 \cdot 2}+\dfrac{1}{x^{1 \cdot 2}}\right)dx$.

4. (i) If a sphere of radius a in. contains water to a depth of x in., the volume is $\int_0^x \pi(2ax-x^2)\,dx$. Simplify this expression.

(ii) Water is poured into a spherical bowl of radius 5 in. at the rate of 4 cu. in. per sec. At what rate is the water-level rising when the depth is 3 inches ?

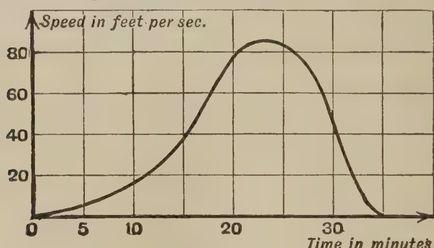

FIG. 64.

5. Fig. 64 represents the time-velocity graph of a train. Construct the distance-time graph and write down the distance travelled, (i) at the end of 15 minutes, (ii) in the whole journey.

L. 2.

1. Simpiify (i) $\frac{d}{dx}(x+2)(x-3)$; (ii) $\frac{dy}{dx}$ if $xy=5$;

(iii) $\int\left(x^2-\frac{1}{x^2}\right)dx$; (iv) $\int_{-1}^{+1}(x+1)^2\,dx$.

2. The horizontal cross-sections of the crater of a volcano are circles with their centres on a vertical line, the radii of the circles at different heights are as follows (measurements in feet):

Height -	0	10	20	30	40	50	60	70	80	90	100
Radius -	0	17	24	27	30	35	41	52	65	78	94

Find its volume in cu. ft. to two significant figures.

3. Two points A, B at the same level are 200 feet apart. A wire fixed at A, B carries a heavy weight at its mid-point O; the portions AO, OB may be taken as straight, but the length of the wire varies with the temperature. When the wire is $2s$ feet long, the depth of O below AB is y feet; when the length is $2(s+h)$ feet, the depth of O is $y+k$ feet. (i) Find a relation between s, y; (ii) find a relation between h, k, s, y; (iii) find a simple form of this relation when h, k are small; (iv) find an expression for $\frac{dy}{ds}$ in terms of s; (v) if the wire stretches from 210 to 210·1 feet, find approximately the increase in depth of O. (C.S.C.)

4. Interpret geometrically $\int_1^4(2x+3)\,dx$; evaluate it (i) by direct calculation and (ii) geometrically.

5. The volume of a bowl of depth x cut from a sphere of radius a is $\pi\int_{a-x}^{a}(a^2-x^2)\,dx$. Compare the volumes of two bowls of depths $\frac{a}{2}$ and $\frac{a}{3}$.

L. 3.

1. The work done in stretching a spring of natural length a inches to a length b inches is $\frac{1}{30}\int_a^b (x-a)\,dx$ foot-lb. Find the work done in stretching the spring from its natural length, 10 inches, to a length of 14 inches.

2. A particle projected in a resisting medium finally comes to rest. It travels s feet in t sec., where $s=50t-\frac{1}{6}t^3$. How far does it go before stopping ?

3. Four rods each of length 10 cm. are jointed together at their ends to form a rhombus ; if the length of one diagonal is x cm., express the area of the rhombus as a function of x, and find (i) the value of x for which the area is greatest, and (ii) the maximum area.

4. The distance of the centre of gravity of a circular cone of height h in. and base-radius r in. from the vertex is

$$\frac{\int_0^h \pi x y^2\,dx}{\int_0^h \pi y^2\,dx} \text{ in.,}$$

where $y=\frac{r}{h}x$. Evaluate this expression.

5. A body moves along a line OA *towards* O in such a way that its velocity at any point is proportional to its distance from O. Its velocity at A, where OA is 30 feet, is 12 feet per sec., and t sec. after passing A, its distance from O is x feet. Find a differential equation connecting x and t, and express $\frac{d^2x}{dt^2}$ in terms of x.

L. 4.

1. (i) Sketch the graph of $y=(1-x)(x-5)$. Calculate the gradient of the tangents at the points where it crosses the x-axis, and find where the gradient is zero. (C.S.C.)

(ii) Find the area of the portion of the curve which lies on the positive side of the x-axis.

2. If a force of P lb. acts on a body for t sec., the impulse given to the body is $\int_0^t P\,.\,dt$ lb.-sec. units. Find the impulse given by a force which acts for t sec. and is such that $P=6-2t$.

3. If the graph of $x^2=5y$ is rotated about the x-axis, find the volume of the portion between $x=1$ and $x=4$.

4. In Fig. 65 the dotted line is a time-velocity graph and the continuous line is a time-distance graph. If the units are properly chosen, can the two graphs both refer to the same motion? Describe the motion in general terms, the time shown being 35 minutes in each case and the maximum velocity 30 miles per hour. Find the distance gone after 14, 28, 35 minutes.

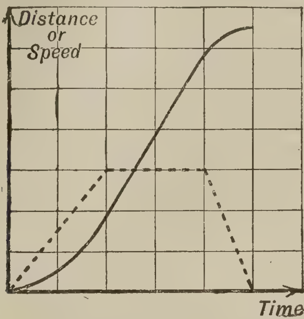

FIG. 65.

5. A weight hangs at the end O of a spiral spring: it is pulled down a certain distance and released, and then oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x inches, the differential equation of the motion is $\frac{d}{dx}(v^2) = -\frac{x}{10}$. (i) If it is pulled down 5″ and released, find its velocity when passing O; (ii) how far must it be pulled down to pass O with a velocity of 1 inch per sec.? (iii) with the data of (i), find an equation expressing δt in terms of x, δx, and write down the integral which gives the time of a complete oscillation.

L. 5.

1. The moment of inertia of a spherical segment, radius r in., height h in., about its axis is $\frac{\pi d}{2}\int_{r-h}^{r}(r^2 - x^2)^2 \,.\, dx$, where d is its density. A sphere of radius r is cut into two segments by a plane dividing a diameter in the ratio $3:1$. Compare the moments of inertia of the two segments about their axes.

2. A piece of wire two feet long is cut into two parts, one of which is bent to form a square and the other an equilateral triangle. If the sum of the areas is a minimum, find the side of the equilateral triangle.

3. (i) For what range of values of x is $2x^3-9x^2$ a decreasing function ?

(ii) Find a minimum value of this function.

4. If a triangle, base b in., height h in., is immersed in water and held in a vertical plane with its base in the surface, the depth of the centre of pressure is given by $\int_0^h by^2\left(\frac{h-y}{h}\right)dy \div \frac{1}{6}bh^2$. Simplify this expression.

5. Fig. 66 is a perspective view of a piece of steel formed by two cylinders of radius a, whose axes are horizontal and cut at right angles. A horizontal plane is drawn at height x above the axes ; what is the shape of the portion of the plane which lies inside both cylinders ? Prove that its area is $4(a^2-x^2)$. Hence prove that the volume common to the two cylinders is $\frac{16}{3}a^3$.

(Trin. Coll.)

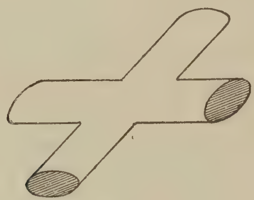

FIG. 66.

L. 6.

1. Find the radius of a cylinder which is cut from a cone of height h in. and base-radius r in. if (i) the volume of the cylinder is a maximum, (ii) the *total* surface is a maximum, given $h > 2r$.

2. The portion of the graph of $y=\frac{x^3}{6}+\frac{1}{2x}$ lying between $x=1$ and $x=2$ is rotated about the x-axis so as to form a surface. The area of this surface is $\int_1^2 2\pi y\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,.\,dx$. Evaluate this expression.

3. The differential equation of a graph is $\frac{d^2y}{dx^2}=4x-6x^2$; the graph cuts the x-axis at the origin and the point (2, 0) ; find the value of y when $x=1$ and the slope at the origin.

4. The gradient of a curve which passes through the origin is given by the equation $\frac{dy}{dx}=x^2(1-x)$; find the area between the curve, the x-axis and the ordinate $x=1$.

5. If a gas occupies v cu. feet under a pressure of p lb. per sq. inch, where $p \cdot v^{1\cdot 42}=300$, the work done in expanding from v_1 cu. ft. to v_2 cu. ft. is $\int_{v_1}^{v_2} p \cdot dv$ ft.-lb.; find the work done in expanding from 5 to 8 cu. ft.

L. 7.

1. (i) Sketch the graph of $y=x(x-1)(x-2)$ for values of x from -1 to $+3$.

(ii) Prove that $\int_0^2 x(x-1)(x-2)dx=0$, and interpret this result geometrically.

(iii) Find the minimum and maximum values of y. (C.S.C.)

2. If a force of P lb. acting on a body moves it s feet in the direction of the force, then the "work done" by the force is $\int_0^s P \cdot ds$ ft.-lb. Find the work done by a variable force $P=3s+2$, which moves a body 10 feet.

3. A train runs from stop to stop in 10 minutes, and its speed is as follows:

Time in minutes -	0	1	2	3	4	5	6	7	8	9	10
Speed in miles per hr.	0	12	24	36	44	49·5	51	50	45	31	0

Use Simpson's rule to find the total length of run. (C.S.C.)

4. (i) At what point on the graph of $y=x^2-2x$ is the gradient 1, if the unit for each axis is 1 inch?

(ii) How is the answer to (i) affected if the unit for the x-axis is 1 inch and the unit for the y-axis 2 inches?

5. If the graph of $y=f(x)$ passes through the origin and has a maximum at $x=1$ and a minimum at $x=2$, but no other turning points, and if the slope at the origin is 1, find a value of $f(x)$.

L. 8.

1. Given $y=(x-1)^3$: (i) find the value of $\frac{dy}{dx}$ when $x=1$; (ii) find the sign of $\frac{dy}{dx}$ when $x=0\cdot 9$ and when $x=1\cdot 1$; (iii) Is $x=1$ a turning point of the function $(x-1)^3$? (iv) find the sign of $\frac{d^2y}{dx^2}$ when $x=0\cdot 9$ and when $x=1\cdot 1$; (v) sketch the graph of $(x-1)^3$ from $x=0$ to $x=2$, and explain the geometrical interpretation of results (i) to (iv).

2. The cubical elasticity of a gas whose volume is v and pressure p is measured by $-v\dfrac{dp}{dv}$. Find the cubical elasticity of a gas for which $p \, . \, v^{1 \cdot 408}=c$, where c is constant.

3. The portion of the curve $y=x^2+1$ between $x=0$ and $x=2$ is rotated about the x-axis to form a solid of revolution. The x-coordinate of the centre of gravity of this solid is

$$\int_0^2 (x^2+1)^2 x \, dx \div \int_0^2 (x^2+1)^2 \, dx.$$

Evaluate this expression.

4. If a horizontal circular disc of radius 4″ is under the influence of a unit charge of electricity at a point 5″ from and vertically above the centre, the density of the induced distribution at a point on the disc x in. from the centre is $\dfrac{3}{2\pi^2} \cdot \dfrac{1}{(x^2+25)\sqrt{16-x^2}}$ units per sq. in. For what value of x is this a minimum ?

5. Given $x^4 \dfrac{d^2y}{dx^2}=1$ and that when $x=1$, $\dfrac{dy}{dx}=1$ and $y=2$, express y in terms of x.

L. 9.

1. The distance of the centre of gravity of a uniform hemisphere of radius a from the centre of the hemisphere is

$$\int_0^a \pi(a^2-x^2)x \, dx \div \int_0^a \pi(a^2-x^2) \, dx.$$

Evaluate this expression.

2. Sketch the graph of $y=\dfrac{1}{x^2}$; mark the point $P(2, \frac{1}{4})$ on it, and draw the tangent at P cutting the x-axis, Ox, at T. Find (i) the gradient of PT, (ii) the length of OT.

3. Interpret geometrically the expressions,

$$\text{(i)} \int_1^2 \frac{1}{x^2} \, dx \, ; \qquad \text{(ii)} \int_1^2 \pi \, . \, \frac{1}{x^4} \, dx,$$

and evaluate them.

4. If $z=\dfrac{1}{\sqrt{y}}$ and $y=49-x^2$, express δz in terms of y, δy, and express δy in terms of x, δx ; hence find δz in terms of x, δx, and deduce the value of $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{49-x^2}}\right)$.

5. Sketch the graph of $(x-2)(5-x)$, and find the area of the portion of it which lies above the x-axis.

L. 10.

1. If a rectangle of height a in. is immersed in water with its plane vertical and its upper edge h in. below the surface, then the depth of the centre of pressure below the surface is

$$\int_0^a (x+h)^2 dx \div \int_0^a (x+h)\, dx.$$

Simplify this expression.

2. A glass is so shaped that when the depth of liquid in it is x inches, the volume of the liquid is $\frac{1}{10}x^4$ cu. in. ; water is poured into it at the rate of 2 cu. in. per sec. ; at what rate is the level rising after 3 seconds ?

3. Given $\dfrac{d^2y}{dx^2}=1-x^3$ and $x=0$ when $y=0$, and $x=2$ when $y=2$, find $\dfrac{dy}{dx}$ when $x=0$, and express y in terms of x.

4. (i) Sketch the graph of $xy=4+x^2$.

(ii) At what points on this graph is the tangent parallel to the x-axis ?

5. The weight per inch length of a rod AB varies as the cube of its distance from A ; the rod is 20 inches long and the weight at B is 5 lb. per inch length. Find (i) the total weight of the rod, (ii) the length of AC if the portions AC, CB are of equal weight.

ANSWERS.

EXERCISE I. a. (p. 3.)

1. -4, 1, 4, 5, 4, 1, -4; (i) 3·24 or $-1·24$; (ii) $3·24 > x > -1·24$; (iii) 5, $x=1$; (iv) 2·87 or $-0·87$; (v) 2, 2·73 or $-0·73$; (vi) 3·45 or $-1·45$.

2. 4·2, 1·8, 0·2, $-0·6$, $-0·6$, 0·2, 1·8, 4·2, 7·4; (i) 0·82 or $-1·82$; (ii) $0·82 > x > -1·82$; (iii) $-0·7$, $x=-0·5$; (iv) 1·56 or $-2·56$; (v) 4, 0·8, 1·44 or $-2·44$; (vi) 0·62 or $-1·62$.

3. 1·15; 0·308; $2 > x > 0$ and < -2; 2·214, $-0·539$, $-1·675$; (*a*) and (*b*) 2·115, $-0·254$, $-1·861$; (*c*) 1·861, 0·254, $-2·115$.

4. $-9·6$, $-3·6$, $-1·2$, $-1·2$, $-2·4$, $-3·6$, $-3·6$, $-1·2$, 4·8; (i) 4·27; (ii) $x > 4·27$; (iii) $-3·83$, $x=2·53$; (iv) $-0·974$, $x=-0·528$; (v) 4·52; (vi) (*a*) 3·78, 0·71, $-1·49$; (*b*) $-2·31$; (*c*) 4·62; (vii) 3, 2, -2.

5. 1, 1·62, $-0·62$; 2·20; 0·66.

6. 2·11, $-0·25$, $-1·86$.

I. b. (p. 7.)

1. 1, 3; $x > 3$ and $x < 1$.

2. No.

3. 2, -3; $2 > x > -3$.

4. 9; 4, -2; $-\infty$, $-\infty$.

7. $x > 10$ or $x < -10$; $0·1 > x > -0·1$.

8. Small, positive; small, negative; large, positive; large, negative.

10. $\sqrt{(25-x^2)}$; circle, centre O; $\pm\sqrt{(25-x^2)}$.

14. Yes, 2·01, 1·999, 1; $2 > x > 1$; no, 1002, -998.

15. 1, 3, 4; $x > 4$ and $3 > x > 1$; 378, -12, -2002; yes, *e.g.* 104.

16. Yes, *e.g.* 2·999 and 3·001; no; 1, 5; 0·3, 0·3 per cent.

18. 98, -102, -6, -201, 199; small, 0·000001; $-0·000001$; $3 > x > 2$ and $1 > x$.

I. c. (p. 12.)

1. $y=\frac{1}{2}(x+4)$.

2. $y=\sqrt{(8x-x^2)}$.

3. $y=\sqrt{(8x-16)}$.

4. $y=\sqrt{(6-x^2)}$.

5. $z=\sqrt{(4x-x^2)}$; $y=\sqrt{\dfrac{x^3}{4-x}}$.

6. $y=\frac{4}{3}\sqrt{(9-x^2)}$. 7. $z(1-x)(3x-1)$. 8. $y=\dfrac{abx}{x^2+b^2}$.

9. $y=\dfrac{d}{x}\sqrt{(d^2-x^2)}$. 10. $x=\dfrac{c(by+az)}{(a+b)(b+c)}$. 11. $V=0{\cdot}0293\,A^{1{\cdot}5}$.

12. $\frac{1}{2}(x^3\sim 1)$.

I. d. (p. 14.)

1. 3, 2, 3, $4a^2+2$, b^6+2. 2. 10, 100, 1, 0·1, 10^{2x}.

3. 3, 0·301, 1·301, 0, $3\log x$. 4. 3, $(x+h)^2-3(x+h)+5$, $\dfrac{1}{x^2}-\dfrac{3}{x}+5$.

5. $2x+h+3$; $2x+3$. 6. 1, $-\dfrac{1}{1+h}$, -1. 7. $9x^2+5$, 2.

8. x^6, $2h^2$.

I. e. (p. 15.)

1. 6, 3, $-\frac{2}{3}$, $-4\frac{1}{2}$, 2, -3, $2c+2$, yes, no.

2. $\frac{3}{5}$, $3\frac{1}{3}$, $4\frac{1}{5}$, -5, 3, 5, $3-\dfrac{6a}{5}$, yes, yes, no.

3. 6, $6\frac{2}{3}$, $\frac{2}{3}$; $\dfrac{2a}{3}+2$, $\frac{2}{3}(a+h)+2$, $\frac{2}{3}$. 4. $\frac{5}{4}$, 1·05, $a+\frac{1}{2}h$.

5. 5, 5, -1; 8, -7; 1 or 3, 6 or -2, 2 (twice); 1·9, $2-h$, $4-2a-h$.

6. (v) Circle, centre origin, radius 5. 7. 12, 9, -3; $-\dfrac{36h}{a(a+h)}$, $-\dfrac{36}{a(a+h)}$.

8. 0, -2, 0; $f(5)$, $f(-5)$. 9. $f(a)$, $f(a+h)$, $\dfrac{f(a+h)-f(a)}{h}$.

II. a. (p. 21.)

1. Yes, *e.g.* 3000; no; 2. 2. 11, no, 2.

3. 0·490, 0·4975; 0·5; no. 4. Approx. 0·09, 0·01, 0·001, 0·0001; 0

5. 1, 7. 6. 1·5, 1·67, 1·91, 1·99; 2; no; $0{\cdot}9991<x<1{\cdot}001$.

7. $\frac{1}{2}(h^2+3h+3)$, 1·5. 8. $16(h+1)$, 16. 9. 5·25, $5-2a$, 2·5.

10. $-\frac{1}{4}$, $-\dfrac{1}{a^2}$, ± 2. 11. $\dfrac{n}{2}$, $\frac{1}{2}$. 12. $\dfrac{50(n-1)}{n}$, 50, 50.

13. $\frac{1}{3}$, $\dfrac{(n+1)(2n+1)}{6n^2}$, $\frac{1}{3}$. 14. $\frac{1}{3}$; $\dfrac{(n-1)(2n-1)}{6n^2}$, $\frac{1}{3}$; $\frac{1}{3}$.

II. b. (p. 26.)

1. 70; 250; 1890 to 1900, 390.

2. $\frac{x}{a}$, $\frac{y}{b}$, $\frac{y-x}{b-a}$, $y-x$ miles per min.

3. 8·4, 24·9, 27·3, 29, 37·2 m.p.h.

4. 1·5, 2·6, 2·25, 3·2, 1·3, 2·4 ft. per sec.

5. 0·3; 1·8; 3; 2·4, 1·2 tons per in.

6. For $t=0$, 5, 10, 15, 20, 25, rate is 1·76, 0·88, 0·43, 0·21, 0·11, 0·05 cu. ft. per sec.

II. c. (p. 30.)

1. 0·47, $-2·9$, 0. **2.** 0·021, $-0·020$, 0·021, 0·009, $-0·0077$.

4. 0·93, 0·38 ins. **5.** 2, 2, 2. **6.** -5. **7.** 0·6.

8. 3 ft., 3 ft. per sec.; 12 ft., 6 ft. per sec., 9 ft. per sec.; 13·23 ft., 12·3 ft. per sec.; $3(2+h)^2$ ft., $12+3h$ ft. per sec.; 12·3, 12·03, 12·000003, 12 ft. per sec.

9. 144 ft., 48 ft. per sec.; 64 ft., 80 ft. per sec.; 134·56 ft., 94·4 ft. per sec.; $16(3-h)^2$ ft., $96-16h$ ft. per sec.; 94·4, 95·84, 95·999984, 96 ft. per sec.

10. 32 ft. per sec. **11.** $32a$ ft. per sec.

12. 23; $3h^2+17h+23$; $3h+17$ ft. per sec.; 17 ft. per sec.

13. 3, 2, 0, -2. **14.** 0·15, 0·18 cm. per sec.

15. 4, $6-h$, $12-2a-h$, $12-2a$, 6, $y=12x-x^2$. **16.** 5·1, 5, $-1·5$.

17. $-\frac{1}{4}$, $-\frac{1}{a^2}$. **18.** $6c^2$. **19.** $\frac{3}{4}$, $\frac{1}{2}$. **20.** m.

21. $4a+b$. **22.** $2ax_1+b$, $-\frac{b}{2a}$. **24.** $2x$. **25.** 0.

III. a. (p. 37.)

1. $5\,.\,\delta x$, 5. **2.** $2x\,.\,\delta x+(\delta x)^2$, $2x$.

3. $10x\,.\,\delta x-3\delta x+5(\delta x)^2$, $10x-3$. **4.** $(1+x+\delta x)^2$, $2x+2$.

5. $1+2x$. **6.** $2x$, $6x$, $14x$, $200x$.

7. $3x^2$, $12x^2$, $15x^2$, $87x^2$. **8.** $12x^3$, $40x^3$.

9. $35x^4$, $5x^4$. **10.** $4\frac{1}{2}x^5$, $-6x^5$, $1-12x^5$.

11. $\underset{\delta x \to 0}{\text{Lt}} \dfrac{(x+\delta x)^n - x^n}{\delta x} = nx^{n-1}$. **12.** $3x^2+4x$, $6x+3x^2$, $2x^2+x$.

13. $3u+5v$. **14.** 3, 5, $30x$, 8, no, yes.

15. Yes, yes, no, no. **16.** $\dfrac{d}{dx}x^n = nx^{n-1}$.

17. $\dfrac{d}{dx}(cx^n) = c\dfrac{d}{dx}(x^n)$. **18.** $\dfrac{d}{dx}(Ax^m + Bx^n) = \dfrac{d}{dx}(Ax^m) + \dfrac{d}{dx}(Bx^n)$.

20. The differentials of x^8, x^{50}, x^{-1}, x^{-3}, x, x°, $x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$, $x^{-\frac{1}{2}}$, x^{-m} are $8x^7$, $50x^{49}$, $-x^{-2}$, $-3x^{-4}$, 1, 0, $\frac{3}{2}x^{\frac{1}{2}}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $-\frac{1}{2}x^{-\frac{3}{2}}$, $-mx^{-m-1}$.

21. 0·41. **22.** $-\frac{1}{30}$. **23.** 0·331, 0·030301. **24.** 0·008. **25.** -20.

III. b. (p. 41.)

1. $7x^6$. **2.** $30x^2$. **3.** $1-\dfrac{1}{x^2}$. **4.** $-\dfrac{5}{x^6}$.

5. $-\dfrac{3}{x^2}$. **6.** $-\dfrac{4}{x^3}$. **7.** $6x-2$. **8.** x^3.

9. $5x^4$. **10.** $375x^2$. **11.** $2x-2$. **12.** $-\dfrac{n}{x^{n+1}}$.

13. a. **14.** ax^{a-1}. **15.** $-\dfrac{3c}{x^4}$. **16.** $\dfrac{b}{2\sqrt{x}}$.

17. $\frac{3}{2}\sqrt{x}$. **18.** $-\dfrac{1}{2x^{\frac{3}{2}}}$. **19.** 0. **20.** $\frac{1}{3}x^{-\frac{2}{3}}$.

21. $2x^2-\frac{1}{5}$. **22.** $\frac{1}{10}-\dfrac{10}{x^2}$. **23.** $21x^2-2$. **24.** $4\pi x^2$.

25. $12x^3-\frac{1}{2}+\dfrac{6}{x^2}$. **26.** $20x^3+6x-\dfrac{1}{x^2}+\dfrac{2}{x^2}$.

27. $3x^2+4x+1$. **28.** $2x-\dfrac{2}{x^3}$. **29.** $2nx^{2n-1}$. **30.** $-kx^{-k-1}$.

31. 6. **32.** 6. **33.** 1, 2. **34.** $42x$.

35. $\mp\frac{1}{4}x^{-\frac{3}{2}}$. **39.** 68, 100; -32, -32. **40.** $392x^7$, $392x^7$.

41. $12+20x$, 5, $2(3+5x)^2$, $60+100x$.

42. $4+6x^2+3x^4$, $2x$, $2+4x^2+3x^4+x^6$, $8x+12x^3+6x^5$.

43. p. **44.** $\frac{125}{36}$.

III. c. (p. 44.)

1. Decr., incr., $-$, $+$, yes. **2.** Incr., incr., $+$, $+$, no.

3. Incr., incr., incr., decr., decr., $+$, $+$, $+$, $-$, $-$, yes.

4. Max. C; min. A, E. **6.** Max.; decr., decr.; $-$, $-$.

7. Min.; incr., incr.; +, +. 8. No; decr., incr.; −, +.
9. No; incr., decr.; +, −. 10. +, −; −, +; −, −; +, +.
11. Yes.
15. AB −, − +; BC +, + −; CD −, − +; DE +, + −; EF −, − +. 18. 1. 19. 3.

III. d. (p. 50.)

1. $x=1$ min.; $x=-2$ min.; $x=0\cdot4$ max.; $x=1$ min., $x=-1$ max.; none; $x=1$ min.; $x=\frac{3}{4}$ min., not $x=0$; $x=1$ min., $x=-\frac{1}{3}$ max.; $x=3$ min., $x=-1$ max.; none.
2. 40,000 sq. ft. 3. 2 cu. ft. 4. 4 ft. 5. 8 in.
6. 80,000 sq. ft. 7. $13\frac{1}{2}$ sq. ft. 8. 12 knots. 9. $10\frac{1}{3}$ sq. ft.
10. $1\frac{3}{4}$ hrs. 11. 18 cu. ins. 12. 0·927. 13. 8 ft.
14. 2·55 cu. ft. 15. 4 ins. from top. 16. 3 ins.
17. 0·184. 18. 8, 4 ins. 19. $\frac{1}{4}$. 20. 1·5 ft.
22. 12 cu. in. 23. 1·68 in. 24. $31\frac{1}{4}$. 25. $\frac{1}{4}$.
26. $\frac{3\sqrt{3}}{16}$. 27. $\sqrt{2}$. 28. 8·16, 7·07 in.
29. On BA produced, $PA=3$ in. 30. $y=\frac{\sqrt{(4x^4+a^2b^2)}}{2x}$, 500, 707 yds.
31. $\frac{1}{2}x(a-2x)(b-2x)$, 1·27 ins.

III. e. (p. 54.)

1. $5(\delta x)^2$, $3x(\delta x)^2+(\delta x)^3$. 2. $x\,.\,\delta x+\frac{1}{2}(\delta x)^2$, $A=\frac{1}{2}x^2$, $\frac{1}{2}(\delta x)^2$.
3. $2\pi r\,.\,\delta r$, 0·63 sq. in. 4. 0·126 cu. in.
5. $\delta t(10-0\cdot6t^2)$, 0·76 ft. 6. $-\frac{c}{p^2}\,.\,\delta p$.
7. $\delta y=\frac{8x}{3l}\,.\,\delta x$; length of wire increases 0·05 in. 8. 540 ft.
9. 210 sq. yd. 10. $\delta M=\frac{\delta k}{10}\left(1-\frac{1}{k^2}\right)$, $-0\cdot12$ tons.
11. $\delta A=x^2\,.\,\delta x$. 12. $\delta V=\frac{1}{4}\pi x^2\,.\,\delta x$.
13. $\delta V=\pi y\,.\,\delta y$. 14. $\delta V=\pi(2ax-x^2)\,.\,\delta x$.
15. $3\,.\,\delta x$; $15(1+3x)^4\,.\,\delta x$; $15(1+3x)^4$.

16. $\dfrac{1}{2\sqrt{z}}\,.\,\delta z$; $2x\,.\,\delta x$; $\delta y \bumpeq \dfrac{x\,.\,\delta x}{\sqrt{(1+x^2)}}$; $\dfrac{x}{\sqrt{(1+x^2)}}$.

17. $\delta y \bumpeq 3(x^3-x+7)^2(3x^2-1)\,.\,\delta x$; $3(x^3-x+7)^2(3x^2-1)$.

18. $\delta y \bumpeq \dfrac{-6\,.\,\delta x}{(3x-5)^3}$; $-\dfrac{6}{(3x-5)^3}$.

19. $\delta u = 2y\,.\,\delta y$; $\dfrac{du}{dx}=2y\dfrac{dy}{dx}$.

20. $\delta V \bumpeq 4\pi r^2\,.\,\delta r$; $\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}$.

III. f. (p. 57.)

1. 0·12 cu. in. per hr.
2. 1·5 in. per sec.
3. 0·637, 0·326 in. per sec.
4. $\frac{11}{6}$ ft. per sec.
5. 0·057 lbs. per sq. in. per min.
6. 0·31 in. per sec.
7. $10^{-10}\times 3V^5$, 3.
8. 0·96, 0, −0·96 sq. in. per min
9. 0·00133 in. per min.
10. 0·242 in. per sec.
11. 0·286 in. per sec.
12. 1·39 sq. cm. per sec.
13. −0·665 mm. per sec.
14. 1·00564.
15. $\dfrac{dQ}{dt}=k(30-Q)$, where k is a constant.

K. 1. (p. 60.)

1. $\frac{17}{500}$, $\frac{1}{25}$, about 0·03.
2. $2x+4$, $-\dfrac{1}{(x-1)^2}$.
3. -2, $3a^2-5$.

5. $y^2=2ax+a^2$, $y=\sqrt{(2ax+a^2)}$.

K. 2. (p. 61.)

1. −1, −1, 0.
2. $4-6x$, $4-6x$, $4-6x$, $\frac{3}{2}+\frac{4}{5}x$, $-\dfrac{6}{x^3}$, $\dfrac{4}{\sqrt{x}}$.
3. 18 ; $-\frac{5}{3}$, $4\frac{1}{12}$.
4. $y=3x-8$.
5. $0·578l$.

K. 3. (p. 61.)

1. $8x-\dfrac{1}{x^2}$, $48x^2-8x+2$, $\frac{5}{3}x^{-\frac{2}{3}}-\frac{1}{3}x^{-\frac{4}{3}}$.
2. $3a^2-12a-15$; 5, −1.
3. 18, 66 ft. per sec.
4. $\dfrac{dx}{dt}=\dfrac{k}{x}$.
5. $1\frac{2}{3}$.

K. 4. (p. 62.)

1. $-\dfrac{2x}{(x^2-2)^2}, \tfrac{1}{2}$. 2. $\sqrt{\dfrac{3}{2x}}, \dfrac{3}{\sqrt{x}}, -\dfrac{12}{x^4}, 2x-\dfrac{2}{x^3}$.

3. $0, \pm 2\sqrt{3}$; ± 2. 4. $\dfrac{d\theta}{dt}=k(15-\theta)$.

K. 5. (p. 63.)

1. (ii) -20000. (iii) No. (iv) $3{\cdot}000$. (v) $\dfrac{9}{(x-1)(x-5)}$, 3.

2. $4-\dfrac{4}{t^2}, 9t^2+\dfrac{4}{t^2}-\dfrac{15}{t^4}, 12{\cdot}5t^{\frac{3}{2}}$. 3. $3x^2-27$; ± 3; $54, -54$.

4. Grows lighter and weight vanishes when $x=7{\cdot}5$.

5. $2x+3+h, 2, 2x+3, 2$.

K. 6. (p. 63.)

1. Not if $x=1$, no, 2. 2. $x=1$ (min.), $x=-3$ (max.).

4. $4x-x^2, x=2$. 5. $s=k\,.\,t^2$.

K. 7. (p. 64.)

1. $-3-6x^2$. 2. $\dfrac{dx}{dt}=2(x+5)$.

3. $h=\dfrac{1200}{\pi r^2}, S=\sqrt{\left(\dfrac{144\times 10^4}{r^2}+\pi^2 r^4\right)}$, 230 sq. ft., $6{\cdot}46$ ft.

4. 81, 0.

5. $\delta y=4x\,.\,\delta x+2\,.\,(\delta x)^2$; $\delta x=3\,.\,\delta z$; $\delta y=12(3z+1)\,.\,\delta z+18\,.\,(\delta z)^2$; $12(3z+1)$.

K. 8. (p. 65.)

1. 2 (min.), -2 (max.). 2. $\tfrac{5}{8}$.

3. $9x^2-7, 18x$; $10x, 10$; $7, 0$; $0, 0$; $-\dfrac{6}{x^2}, \dfrac{12}{x^3}$.

4. ± 125. 5. $60x-\pi x^2$, $9{\cdot}55$ in.

K. 9. (p. 65.)

2. $x\sqrt{(25-x^2)}$, $12\frac{1}{2}$ sq. cm.

3. $\frac{1}{100}$, $\frac{7}{10000}$, $\frac{2+h}{300}$, $\frac{1}{150}$.

4. $\frac{dy}{dx}+, -, -, +$; $\frac{d^2y}{dx^2}-, -, +, +$.

5. 0·87 cu. ft.

K. 10. (p. 66.)

1. 0·00101, 0·00107, 0·00112, 0·00107.

2. $2x+\frac{1}{x^2}$, $\frac{2}{3}\sqrt[3]{\frac{2}{x}}$.

3. 12·3.

4. Positive if $0<x<4{\cdot}76$, increasing if $x<3$; $x=3$.

5. 8 ft., 48 sq. ft., 256 cu. ft.

K. 11. (p. 66.)

1. 3400, 3564, 3417·84 ; 164, 178·4 ; $180-16h$; 180 ft. per sec.

2. Each $\simeq 0{\cdot}43$, $\frac{d}{dx}\log x \simeq \frac{0{\cdot}43}{x}$.

4. $\frac{1}{2}$.

5. 8 in.

K. 12. (p. 67.)

1. $-\frac{5}{x^2}$, $-\frac{x^2}{5}$, 1.

2. $\frac{1}{25}$ when $y=+10$.

3. $2x+h-3$, $2x-3$, 1·5.

4. $\frac{2x}{(1-x^2)^2}$.

5. 75 ; 1·2, $-1{\cdot}2$; 30 (min.), -30 (max.).

IV. a. (p. 69.)

1. $y=x^4+c$.

2. $y=\frac{1}{5}x^5+\frac{1}{2}x^2+c$.

3. $\frac{dy}{dx}=2x$, parallel curves.

4. $y=\frac{1}{12}x^4+ax+b$.

5. $y=\frac{1}{2}x^2+c$.

6. $y=x^3-x^2+c$.

7. $y=4x+c$.

8. $y=ax+b$.

9. $5x^4$, $y=\frac{1}{5}x^5+c$, $y=\frac{3}{5}x^5+c$.

10. $8x^7$, $y=\frac{1}{8}x^8+c$, $y=\frac{5}{8}x^8-2x+c$.

11. $-\frac{1}{x^2}$, $y=-\frac{1}{x}+c$, $y=-\frac{4}{x}+c$.

12. $-\frac{4}{x^5}$, $y=-\frac{1}{4x^4}+c$, $y=x+\frac{3}{4x^4}+c$.

13. $\frac{1}{2\sqrt{x}}$, $y=2\sqrt{x}+c$, $y=\frac{1}{2}x^2+6\sqrt{x}+c$.

14. $y=\frac{5}{3}x^3-\frac{3}{2}x^2+9x+c$.

15. $y = \frac{1}{6}x^6 - \frac{1}{3}x^3 + 3x + c.$ 16. $y = \frac{1}{4}x^4 - \frac{1}{2x^2} + c.$

17. $y = \frac{10}{3}x^{\frac{3}{2}} + c.$ 18. $y = -\frac{3}{x} - \frac{5}{2x^2} + c.$

19. $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x + c.$ 20. $y = \frac{1}{12}x^4 - \frac{1}{6}x^3 + ax + b.$

21. $y = \frac{3}{101}x^{101} - \frac{5}{99x^{99}} + c.$ 22. $c = -1.$

23. $s = 12 + 50t - 16t^2.$ 24. $s = \frac{2}{3}t^3 + 5t + 8.$

25. 165 ft. 26. $y = x^3 - x + 3.$

27. 0 ; 0 ; $y = \frac{0{\cdot}0001}{12}x(20 - x)(400 + 20x - x^2)$, 0 ; $\frac{1}{15}$, $-\frac{1}{15}$; 5 in.

28. 10,000 ft. lb.

29. $x = 12t$, $y = 16t^2$; $y = \frac{1}{9}x^2$; $2\frac{1}{2}$ sec. ; 30 ft. ; $\frac{20}{3}$; $x = 10$, $y = \frac{100}{9}$.

30. $\frac{3}{4}$ in. per sec., 8 in. 31. $\frac{dT}{dx} = a(K - T).$ 32. $\frac{dp}{dx} = -ap.$

IV. b. (p. 75.)

1. $\frac{1}{3}x^3$, $2\frac{1}{3}$, $\int_1^2 x^2 dx.$ 2. $\frac{2}{3}x^{\frac{3}{2}}.$ 3. 4.

4. 1, 4 ; $4\frac{1}{2}$. 5. $\frac{1}{2}$, 0·83. 7. $7\frac{1}{3}$, $\frac{3}{32}$, $-\frac{2}{3}$, $\frac{19}{24}$.

8. πy^2, $\pi(y + \delta y)^2$, $\frac{1}{5}\pi x^5$, 0·148. 9. $\frac{1}{27}\pi x^3$. 10. 1·57.

11. $\sqrt{(100 - x^2)}$, $\pi(100 - x^2)$, $\pi(100 - x^2) \,.\, \delta x$, $\pi(100x - \frac{1}{3}x^3)$,
$\frac{\pi}{3}(2000 - 300x + x^3)$, 2090 cu. in.

12. 47·4 cu. in. 13. 14·2 cu. ft. 14. $2\frac{2}{3}$. 15. $\frac{23}{48}$.

16. 0. 17. Each $= 8\frac{5}{6}$. 18. 2, $\frac{5}{8}$ in. 19. 1·56 in.

20. $\frac{1}{2}Wr^2$. 21. $\frac{2}{5}Wr^2$. 22. $2\frac{1}{7}$. 23. $1\frac{7}{8}$.

24. 210 ft. lb. 25. $\frac{31}{56}$. 26. $\frac{9}{16}$. 27. $4\frac{2}{7}$.

28. $\pi\rho a^3$. 29. 90 lb. 31. 2520 cu. ft. 32. 3050 cu. in.

IV. c. (p. 82.)

1. $\frac{1}{3}$, $\frac{1}{3}$. 2. 120·2, 118·7.

3. 38·687, 39·270 ; 1·4 per cent. 4. 41·59, 0·02 per cent.

5. 3·28 mi. 6. $10^5 \times 3{\cdot}040$ cu. ft. 7. 1030 ft.

8. 2094, 2094 cu. cm. 9. 0·1678. 10. 127·5.

L. 1. (p. 84.)

1. 0·26.

2. $y = \frac{1}{6}(3x^2 - 4x^{\frac{3}{2}} + 7)$, $y = 4x + 1 + \dfrac{1}{2x}$.

3. $\dfrac{5}{3}x^3 - \dfrac{4}{x} + c$; $\dfrac{5}{11}x^{2\cdot2} - \dfrac{5}{x^{0\cdot2}} + c$.

4. $\pi x^2\left(a - \dfrac{x}{3}\right)$, $\dfrac{4}{21\pi} = 0\cdot0606$ in. per sec.

5. 4800, 29500 yds.

L. 2. (p. 85.)

1. $2x - 1$, $-\dfrac{5}{x^2}$, $\dfrac{x^3}{3} + \dfrac{1}{x} + c$, $2\frac{2}{3}$.

2. $10^5 \times 7\cdot09$.

3. $ky = hs$, $\dfrac{s}{\sqrt{(s^2 - 10000)}}$, 2 in.

4. 24.

5. 135 : 64.

L. 3. (p. 86.)

1. $\frac{4}{15}$.

2. $333\frac{1}{3}$ ft.

3. $\dfrac{x}{2}\sqrt{(400 - x^2)}$, $10\sqrt{2}$, 100.

4. $\frac{3}{4}h$.

5. $\dfrac{dx}{dt} = -0\cdot4x$, $0\cdot16x$.

L. 4. (p. 86.)

1. 4, -4; $x = 3$; $10\frac{2}{3}$.

2. $6t - t^2$ lb. sec.

3. $\dfrac{1024\pi}{125} = 25\cdot7$.

4. Yes; 6160, 18480, 21560 yd.

5. 1·12 in. per sec., 4·47 in., $2\displaystyle\int_{-5}^{5}\sqrt{\left(\frac{20}{25 - x^2}\right)}\,dx$.

L. 5. (p. 87.)

1. 459 : 53.

2. 4·52 in.

3. $0 < x < 3$, -27.

4. $\frac{1}{2}h$.

5. A square.

L. 6. (p. 88.)

1. $\dfrac{2r}{3}$, $\dfrac{rh}{2(h - r)}$ in.

2. $\dfrac{47\pi}{16} = 9\cdot23$.

3. 1·5, $\frac{1}{4}$.

4. $\frac{1}{30}$.

5. 65·1 ft. lb.

L. 7. (p. 89.)

1. $\pm\frac{2\sqrt{3}}{9}=\pm 0{\cdot}385$.
2. 170 ft. lb.
3. 5·79 mi.
4. $\frac{3}{2}, -\frac{3}{4}; \frac{5}{4}, -\frac{5}{16}$.
5. $\frac{x}{12}(2x^2-9x+12)$.

L. 8. (p. 89.)

1. 0; +, +; no; −, +.
2. $1{\cdot}408p$.
3. $\frac{155}{103}=1{\cdot}50$.
4. $\frac{1}{3}\sqrt{21}=1{\cdot}53$.
5. $y=\frac{1}{6}\left(\frac{1}{x^2}+8x+3\right)$.

L. 9. (p. 90.)

1. $\frac{3a}{8}$.
2. $-\frac{1}{4}, 3$.
3. $\frac{1}{2}, \frac{7\pi}{24}=0{\cdot}916$.
4. $\delta z \triangleq -\frac{\delta y}{2y^{\frac{3}{2}}}$, $\delta y \triangleq -2x\,\delta x$, $\frac{x}{(49-x^2)^{\frac{3}{2}}}$.
5. $4\frac{1}{2}$.

L. 10. (p. 91.)

1. $\frac{2(a^2+3ah+3h^2)}{3(a+2h)}$.
2. 0·232 in. per sec.
3. 0·8, $y=\frac{1}{20}(16x+10x^2-x^5)$.
4. (2, 4); (−2, −4).
5. 25 lb.; 16·8 in.

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